IESE Business School

University of Navarra

# *Occasional Paper*

OP no 03/2

September, 2002

# ON THE FOUNDATIONS OF A UNIFIED THEORY INCLUDING SET THEORY, NON-STANDARD ANALYSIS AND FINITE ANALYSIS

Miguel Soler*

* Professor of Finance, IESE



**IESE Business School - Universidad de Navarra**
Avda. Pearson, 21 - 08034 Barcelona. Tel.: (+34) 93 253 42 00 Fax: (+34) 93 253 43 43
Camino del Cerro del Águila, 3 (Ctra. de Castilla, km. 5,180) - 28023 Madrid. Tel.: (+34) 91 357 08 09 Fax: (+34) 91 357 29 13

# ON THE FOUNDATIONS OF A UNIFIED THEORY INCLUDING SET THEORY, NON-STANDARD ANALYSIS AND FINITE ANALYSIS


## Abstract

The paper shows how the principle that the whole must be greater than the part is not necessarily inconsistent with being bijected with a proper subset, provided that equicardinality is reinterpreted as related with definability and not with sameness of size. An explanation for such reinterpretation is offered on the basis of availability, which leads to the problem of graduality, as raised by the Sorites Paradox, and to Finite Analysis, as developed by S. Lavine from the finiteness theorems of J. Mycielski.

In order to have a formal version of availability, non-standard analysis is then considered: first in its classic, model theoretic version, showing that Robinson's infinities are not Dedekind's, and later in its axiomatic version, E. Nelson's Internal Set Theory (IST), which can be used not only to prove the existence of infinitesimals but also, as shown in this paper, to deduce many properties about infinity, proving as a theorem ZFC's Axiom of Infinity. IST is shown to contain standard "natural" numbers with non-standard predecessors that must be considered (Dedekind) infinite, requiring a redefinition of infinitesimals and their inverses.

With such redefinition, IST is compatible with J. H. Conway's surreal numbers, providing a foundation for a unified number system including the traditional Cantorian transfinite ordinals as well as their infinitesimal inverses. The paper argues why the new number system should be used inside IST–AI, instead of inside ZFC. Such number system is shown, to conclude, as being able to provide a measure for the size of any kind of sets, the cardinals no longer providing such a measure under the proposed reinterpretation.



**Keywords**: number system, set theory, non-standard analysis, internal set theory, finite analysis.

# ON THE FOUNDATIONS OF A UNIFIED THEORY INCLUDING SET THEORY, NON-STANDARD ANALYSIS AND FINITE ANALYSIS

## 1. Two disjoint number systems? a parallel between physics and mathematics

The major puzzle of present-day Physics is the well known problem that its two main theories, Relativity and Quantum Mechanics, are two disjoint theories: the explanation that is valid for the large entities of the Universe cannot be applied to the small ones, and vice versa. Although it is not seen as problematic, a parallel situation is found in Mathematics: the theory that is used to talk about large entities, Set Theory, with its transfinite sets, is disjoint to the theory that gives a foundation to the smallest mathematical entities, Non-Standard Analysis, with its infinitesimals; with two aggravating circumstances here: that Set Theory is intended to be the foundation for all mathematics, and that Non-Standard Analysis contains its own infinite numbers (the inverses of infinitesimals).

In [Stroyan 1991, p. 207] we are told that: “Cantor’s infinite numbers violate Leibniz’s Principle [stating, roughly, that *ideal numbers* were to have the same properties as *finite numbers*], so that the infinite numbers of Robinson are a completely disjoint theory, though this fact certainly implies no conflict. The great progress in measure theory and topology no doubt has contributed to modern disfavor for infinitesimals, but this success does not imply that there is only one kind of infinite number. We hope that the mathematical community will recognize the existence of (at least) two kinds of infinite numbers.”

Would you be happy with two completely disjoint kinds of natural numbers? I bet you would not. Or can you imagine counting with two different kinds of natural numbers and obtaining different results? Neither should you admit two different kinds of infinite numbers. I also depart from Stroyan’s view that Cantor’s transfinite numbers are much more important than Robinson’s infinitesimals: from the non-negligible point of view of their applicability to real life, Integral and Differential Calculus are much more widely used than Set Theory, and infinitesimals are an important part of Calculus.

As in Physics, a unified theory encompassing both the very large and the very small should be highly welcome in Mathematics. Such a theory should have the infinitesimals as inverses of the transfinite numbers. A side advantage would be that, as the main use of non-standard analysis is to provide a justification for infinitesimals, it would also justify the use of their infinite inverses, whereas in set theory an ad hoc axiom is needed in order to justify the existence of infinite sets. In addition, Cantor’s transfinite numbers do not allow us to reproduce the surprising results obtained, mainly by Euler, using infinities jointly with infinitesimals in a naïve manner, like the sums of infinite series, or the results of infinite products, of the kind:

$$\zeta(2) = 1 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots = (1/1\text{-}1/2^2)(1/1\text{-}1/3^2)(1/1\text{-}1/5^2) \ldots = \pi^2/6$$

We have avoided the errors committed by Euler, with his indiscriminate use of infinity, at the cost of forgoing his powerful results.

Another parallel with Physics is the appearance of "dual" theories, theories that refer to different entities but that are formally equivalent, to the point where a demonstration in one theory is valid in the other (duly "translated" to the equivalent, or "dual", entities of the other theory). One such is the "T-duality" or symmetry between the size of two different collapsed dimensions, according to which a super-string in a dimension rolled with radius R has the same properties, and produces the same particles, as another one in a dimension rolled with radius 1/R, meaning that a very small dimension is equivalent to a very large one. A similar situation appears in Mathematics between the rational numbers (excluding 0) and their inverses, which form an Abelian group, with a $1/x$ for each $x$. Should there not be the same symmetry between the infinitesimal and the transfinite?

Considering Set Theory as an extension of Logic to the quantitative realm, duality in the former must be considered as a consequence of its use in the latter. One of the classical sections of Modal Logic is the Theory of Duality [Jansana 1990, chapter 12], where "duality" is identified with the formal notion of "isomorphism", proving [Jansana 1990, proposition 8.4, pp. 115-6] that: "If two frames are isomorphic (...) the same modal formulas are valid for them, in other words, the same logic holds for them."

In Mathematics a striking duality has been shown in [Mycielski 1986]: that any theory with infinite elements has a dual finite version with equivalent entities that are actually finite and, at most, only potentially infinite. Such duality, which will be considered later, forces us to review the very foundations of infinity.

## 2. Bijectability reinterpreted: three solutions to Galileo's Paradox

The (also called) Reflexivity Paradox has been known since ancient Greece[i]: that in an infinite set it is possible to pair its elements with those of a proper subset (let us abridge this property as "**self-pairing**"); for example, the naturals can be paired onto the even numbers:

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\
2 & 4 & 6 & 8 & 10 &
\end{array}
$$

the paradox consisting in that two well established principles come into conflict:

1st Principle: The whole is bigger than a (proper) part.

2nd Principle: Two collections whose elements can be paired have the same size.

Three solutions to this conflict are possible:

I) The ***Aristotelian*** or classical solution: the two principles being "obviously" true, the conflict *demonstrates* that actual quantitative infinities are impossible (unlimited collections are only "potentially" infinite, they do not have a definite size).

II) The ***Cantorian*** or set-theoretical solution: there can be actually infinite sets that do really have the same number of elements as some of their proper subsets: the 1st Principle fails.

III) The ***Euclidian*** or neo-classical solution: in infinite sets the 2nd Principle fails; they can be paired in spite of having a different size (as is the case, in particular, between a set and a proper subset).

I have adopted the names from [Mayberry 2000][ii], where it is proposed, inter alia, that the Axiom of Infinity, usual in ZFC, be replaced with a new *Axiom of Euclidean Finiteness*, which is intended to force inside the theory a stricter version of the Euclidean requirement: the condition that the whole must be greater than the part, prohibiting its pairing.

Although Mayberry's approach is very appealing, **set theory can be formally developed just as well under the Euclidian as under the Cantorian solution**; only the Aristotelian solution precludes it. What is relevant for ZFC is that a set being self-paired is not contradictory; it was Dedekind (following a suggestion from Bolzano) who first used such controversial property to define an infinite set: "a set is *(Dedekind) infinite* if it can be self-paired". The interpretation of such (Cantorian or Euclidean) property matters only in the meta-theory.

Pairing is also used to compare infinite sets: two sets are said to have the same "cardinal" (to have equicardinality) if they can be bijected. "Cardinal" means "quantity", but the name by itself does not imply that two sets can be paired only if they have the same number of elements, i.e. the same size (the Cantorian solution): two sets would equally be said, by that definition, to have the same cardinal if it was possible to pair them in spite of being of different size (if this was a possibility for infinite sets, the Euclidean solution).

## 3. Equicardinality as equidefinability

However, if being bijectable is not to be interpreted as having the same number of elements, what will happen to the Theory of Transfinite Cardinals (including the inaccessible ones, the *large cardinals*)? Formally, nothing. The Euclidean solution affects only the interpretation in the meta-theory, not the formal theory: insofar as we do not change the axioms, all the theorems will remain valid.

As said in [Delahaye 2000, p. 38]: "In a manner that has surprised the mathematicians, the axioms of large cardinals appear in a well defined linear order as if they designed a real hierarchy of infinities". But if the large cardinals' hierarchy is not about increasing size, what is the property that is being hierarchised? A solution can be provided by an alternative (meta) proof of the fundamental Cantor's Theorem proving the existence of non-denumerable sets. Let us recall that Cantor's demonstration that the set of all the natural numbers, $\omega$, cannot be paired with $\mathbf{P}\omega$ (the set of all the subsets of $\omega$) is based on the possibility of defining, on the basis of a hypothetical pairing, a subset that is "the set of all the numbers that do not belong to its correlate", which converts it, when applied to its own correlate, into a variant of the Liar Paradox (does the correlate of the set of all the numbers that do not belong to its correlate belong to such set – its correlate?)[iii]. Subsequently, either the pairing is not possible or the theory is contradictory because it admits the formulation of a Liar Paradox (a conclusion parallel to Gödel's Incompleteness Theorem).

The alternative (meta) proof can be formulated on the basis of the impossibility of having a finite definition for all the elements of **P**ω (the set of all subsets of ω): any finite subset of ω can be finitely defined, at least by its *extensional* definition, i.e. the listing of all its members. The extensional definition is not possible for the infinite subsets of ω if we limit ourselves to a finite (although indefinitely large) language: only finite *intensional* (through a finite formula) definitions will be available and, therefore, many subsets will be undefined, because **P**ω is a combinatorial set in which any random combination of numbers is admitted, and a random infinite series of numbers cannot, by the definition of randomness, be described by a finite formula. However, if we could define a pairing (we only admit finitely defined pairings) between the natural numbers and the subsets of ω, as any natural number is finitely definable (extensionally at least), the defined pairing would provide a finite definition for each subset (the combination of the definition of the correlate number with the definition of the pairing): as it is impossible, as said, to have a definition for all the subsets, such a defined pairing is not possible.

Let us call two sets *equidefinable* if the elements of each of them are finitely and uniquely definable in terms of the elements of the other. If two sets can be paired (defining the correlate for each element in both senses), the elements of each one can be defined in terms of the elements of the other. Conversely, if all the elements of one set are finitely and uniquely (that is, well) defined in terms of the elements of another, and vice versa, such a definition will be, because of the uniqueness, a one-to-one function between both sets. The above demonstrates that:

"Two sets are equipotent (can be paired) if, and only if, they are equidefinable"

Correspondingly, a set will be *maxipotent* to another if the elements of the latter can be defined in terms of the elements of a subset of the former and not vice versa.

According to these equivalencies between potency and definability, the hierarchy that the transfinite cardinals introduce inside the ordinals may be reinterpreted as the increasing levels of ordinals which are equidefinable, not of ordinals that have the same size, understood as number of elements.

## 4. Is there a real transfinite cardinals arithmetic?

It is worth recalling that Cardinal Arithmetic is exceedingly poor: addition does not allow us to obtain cardinals bigger than the biggest that we already have, and the same has to happen with the repetition of addition, multiplication, and subsequently, with exponentiation *if it was to be defined as repetition of multiplication* (it is not). None of these operations can give us any new cardinal bigger than the biggest of its terms.

In Ordinal Arithmetic, as in elemental Arithmetic, there is one basic operation: successor, $x' = x \cup \{x\}$; addition merely abridges the repetition of the successor operation ($m + n = m''' _{\dots n \text{ times}}\, '$); therefore multiplication abridges the repetition of addition ($m$ x $n = m + m + _{\dots n \text{ times}} + m$); and once we can have repetition of repetitions (that is, if we have multiplication besides addition), then we can have exponentiation as the repetition of multiplication ($m^n = m$ x $m$ x $_{\dots n \text{ times}}$ x $m$)[iv].

In contrast, no basic operation is defined for the infinite cardinals, in the absence of the Generalized Continuum Hypothesis (GCH), that could take us from some cardinal, $\aleph_\alpha$ , to the following cardinal, $\aleph_{\alpha+1}$ or $\aleph_\alpha^+$. With the GCH we have

$$\aleph_\alpha^+ = |\mathbf{P}\ \aleph_\alpha|$$

but if we try to define addition as a repetition of this "successor operation"

$$\aleph_\alpha\ (+)\ \aleph_\beta = \aleph_\alpha^{+ \ldots (\aleph_\beta \text{times}) \ldots +} = |\mathbf{P} \ldots (\aleph_{\beta \text{ times}}) \ldots \mathbf{P}\ \aleph_\alpha| = \aleph_{\alpha+\beta}$$

we obtain cardinals much larger than one would expect from the intuitive concept of addition[v]. Presumably because of this, Cantor defined, in 1887, the addition of two cardinals, $\mu$ and $\nu$, as "the cardinal of the union of two disjoint sets having their cardinalities" (from [Mosterin 1980, pp. 257-63]), that is, taking $a$ and $b$ such that $|a| = \mu$, $|b| = \nu$ and $a \cap b = \varnothing$, cardinal addition was defined as:

$$\mu\ (+)\ \nu = |a \cup b|$$

a definition that is still used (e.g. in [Jech 1978, p. 24] ); a case of such disjoint sets can be the cartesian products, $\mu \times \{0\}$ and $\nu \times \{1\}$, as used in the definitions in [Kunen 1990, p. 28] or [Mosterin 1980, definition 20.1]:

$$\mu\ (+)\ \nu = |(\mu \times \{0\}) \cup (\nu \times \{1\})\}|$$

but $\mu$ and $\nu$ being ordinals imply that $\mu \subset \nu$ or $\nu \subset \mu$, and, subsequently, either $\mu \cup \nu = \mu$ or $\mu \cup \nu = \nu$, and then, inevitably:

$$\mu\ (+)\ \nu = \max(\mu, \nu)$$

so defined, addition cannot give us any new cardinal. Anyway, this "operation" is called "addition" because, I presume, in the finite case it gives the same sums as ordinal addition (but that will also happen defining $\aleph_\alpha\ (+)\ \aleph_\beta = \aleph_{\alpha+\beta}$ ), which is due to the fact that cardinal addition is equivalent to the cardinal of ordinal addition, according to a theorem [Mosterin 1980, theorem 20.8]:

$$\mu\ (+)\ \nu = |\mu + \nu|$$

Using a similar property and in spite of Cantor having, in 1887, defined cardinal multiplication as "the cardinal of the cartesian product", $|\mu \times \nu|$, cardinal multiplication "(.)" is usually defined directly (e.g. [Kunen 1990, 10.9 (2) ] or [Jech 1978, p. 24] ) as "the cardinal of ordinal multiplication",

$$\mu\ (.)\ \nu = |\mu \,.\, \nu|$$

which is equivalent (as shown in [Mosterin 1980, theorem 20.21]), implying that multiplication is the repetition of addition because (applying repeatedly theorem 20.8 from [Mosterin 1980] ):

$$\mu\ (.)\ \nu = |\mu \,.\nu| = |\mu + _{\ldots (\nu \text{ times}) \ldots} + \mu| = \mu\ (+) _{\ldots (\nu \text{ times}) \ldots} (+)\ \mu$$

but such repetition implies that multiplication cannot give us any new cardinal either and, effectively, we have again [Jech 1978, corollary 3.11]:

$$\mu \, (.) \, \nu = \max(\mu , \nu)$$

and the same would happen with exponentiation, $\mu^{(\nu)}$, if it were to be defined as the repetition of multiplication,

$$\mu^{(\nu)} = \mu \, (.) \, ^{\cdots}{}_{(\nu \text{ times})}{}^{\cdots} \, (.) \, \mu = \left| \mu \, . \, {}^{\cdots}{}_{(\nu \text{ times})}{}^{\cdots} \, . \, \mu \right| = \left| \mu^{\nu} \right|$$

Because of this, Cantor defined, in 1895 [Mosterin 1980, p. 261], an "operation", which he *called* "exponentiation" (despite the fact that it is not the repetition of the product, although in the case of finite ordinals it coincides with ordinal exponentiation), that can produce a cardinal bigger than its terms: "the cardinal of the set of all the functions that have the exponent, ν, as domain and the base, μ, as range" (a set frequently designated as ${}^{\nu}\mu$ ); in a modern version [Kunen 1990, combining definitions 10.24 and 10.25]:

$$\mu^{(\nu)} = \left| \{ f : f \text{ is a function} \wedge \text{domain } (f) = \nu \ \wedge \ \text{range } (f) \subset \mu \ \} \right|$$

this is an *ad hoc* definition which, in the case that $\aleph_\alpha < \aleph_\beta$ , gives precisely:

$$\aleph_\alpha{}^{(\aleph_\beta)} = 2^{(\aleph_\beta)} = \left| \mathbf{P} \, \aleph_\beta \right|$$

This definition is not equivalent to the other because, for example, $|{}^{\omega}2| = |\mathbf{P}\omega| > \omega$, while $|2^{\omega}| = |2.2. \cdots_{(\omega \text{ times})}| = \omega$ . Cantor's cardinal exponentiation is not equivalent to the repetition of his multiplication: it is not, then, real exponentiation. Additionally, we do not have a "calculation" that shows us, for example, that $2^{(\omega)} = |\mathbf{P}\omega|$, but a "definition" that *calls* "$2^{(\omega)}$" the cardinal of a set equivalent to **P**ω. Subsequently, the only operation of Cardinal Arithmetic that can give us new transfinite numbers may hardly be called an "operation".

If cardinal numbers do not measure size, what size does **P**ω really have? In order to know, we need to *introduce some additional number system* that is really capable of measuring quantities of elements in a set. But we should first show why and how two sets could be paired without having the same number of elements.

## 5. Self-pairing and unavailable numbers

Before seeing formally, with the help of complementary theories, how pairing between sets of different size could be possible, let us consider an intuitive explanation. (Notice that, on the contrary, except for infinite sets being "peculiar", **no "mechanism" has been suggested to explain how it could be possible for the part to be as great as the whole**, contradicting what is philosophically considered an *obvious* principle).

Apparently, the Cantorian and the Eulerian solutions could be formally distinguished because a function of, say, the even numbers *into* the natural would be *onto* (each natural would be actually paired with an even) in the Cantorian solution, and *not onto* (not every natural would have an even pair) in the Euclidian. In fact, set theory could well not provide a means to know which is the case if infinite sets were to have some property that allowed them to appear as paired in spite of their relation not being really onto; in fact, a set should be considered (Dedekind) infinite if a pairing between its elements and those of a proper subset can be found for all its *available* (a naïf concept, for the moment) elements, in spite of its not being known whether the pairing is valid for all of its elements, including the unavailable ones.

Is it true inside the first 1,000 natural numbers that “for each *n* there is a *2n*”? It is... for the first 500: we need to look at the 501st to find a number that does not comply with the formula. Looking up to the second half of a numerical series is one way to check such incompliance, but for very large numbers that would be very annoying (and not possible in practice); an alternative and much more practical way is to reason that, if *m* is the largest, or last, number in the series, then the *m/2* +1 member of the series cannot have a correlate even number, as *m* + 2 is outside the series. But in an infinite series there is no such last number, *m,* so we cannot check what happens with the corresponding *m/2* +1, neither is there a “second half” to look at, but we believe the formula to be valid for all *n* inside it.

Let us now consider what happens with “unavailable” natural numbers, numbers that are so *large* –we continue being naïf– that they will never be accessible for any mathematician in our Universe (as it is limited in size, time and complexity, there is no doubt that such numbers exist): if *n* is an available number, so should be *n* +1, as it is only slightly larger, and therefore 2*n* must be also available; subsequently, if *v* is unavailable, so will be *v/2* and *v/2* +1; as a consequence, that “for each *n* there is a 2*n*” will be true for any available predecessor of *v*, but then we could say that *v*, expected to be a finite natural number, can be paired with the series of its even predecessors, as we cannot show (since none is available) any number before *v* that does not satisfy the pairing.

That the successor of any available number is also available and the predecessor of any unavailable is also unavailable implies that numbers must change gradually from available to unavailable, the meaning of graduality being shown by the following paradox.

## 6. The Sorites Paradox: is infiniteness gradual?

Many mathematical reasonings are based on “induction”, for example:

(I) 0 is finite;
(II) if *n* is finite so is n + 1;[vi]
(III) subsequently, all numbers are finite.

No matter how many times we add “+1”, we will never attain an infinite natural number, even after repeating such sum an infinite number of times (as needed in order to have all the elements of ω).

(I) and (II) seem to be equally true if we substitute “available” in place of “finite”, leading to the conclusion: (III) “all numbers are available”: but that is in disagreement with our knowledge, as we know that, although the “first” numbers are available, only a finite amount of natural numbers are available, there must be an infinity of unavailable numbers. Where does inductive reasoning fail with “available”? And, does the failure apply also to “finite”?

A classic case of failure of inductive reasoning can be found in the Sorites Paradox, a modern version of which (the classic being about a heap –*soros*– of millet or about a bald man) may be formulated as follows:

(I’) A man with 0 $ is poor;
(II’) if a man with n $ is poor, he continues being poor after receiving 1 $;
(III’) subsequently, no matter how many times a poor man receives 1 $, he remains always poor.

Now (I') is undoubtedly true, and (III') is undoubtedly false if the man receives, for example, a billion times 1 $: therefore, either induction fails or (II') is false. But (II') cannot be (completely) false, because in that case its contrary, "a poor man becomes rich if he receives 1 $", would be (completely) true, which it is not: (II') can be, at most, partly false (otherwise induction must fail), but being "partly false" seems to be against the *Principle of Bivalence*: either a sentence is true or it is not true (false), it cannot be *partly* true.

In [Sainsbury 1995, pp. 23-51] three, commonly mentioned solutions to this conflict are considered:

1) The *epistemic* solution: says that the paradox is due to ignorance, that there is a *frontier* number $f$, unknown to us, such that a man with $f$ $ is poor and with $f$+1 $ is rich. (II') is then false in the case $n = f$ , induction does not fail.

2) The *supervaluations* solution: there is, in these gradual cases, a *penumbra* zone inside which it cannot be said, for example, that a man is poor, nor can it be said that he is not poor. (This is not the same as saying that he is "neither poor nor not poor", because this is equivalent to being "poor and not poor", a contradiction which does not need to happen with supervaluations). It simply happens that, within the penumbra zone, the concept is not well enough defined to be decided. Induction fails along the penumbra zone: the conclusion before the penumbra cannot be applied after it. A problem with this solution is "higher order vagueness": the penumbra itself must have a vague frontier, the zones within which the penumbra begins and end, and so forth.

3) The *degrees of truth* solution[vii]: what fails is no less than Bivalence, (II') is partly true (and partly false); if a man needs to have, say, 10,000 $ to be not poor, then (II') is 1/10,000 false; after receiving 10,000 times 1 $ the conclusion would be completely false, partly false with lesser times. This solution can be easily combined with the second one: the leakage taking place only inside the penumbra.

The epistemic solution has many defenders as it preserves bivalence and induction, but seems contrary to our use of "available" as a *vague* concept, one that, like poor, tall, big, old, heap, bold, red, etc. does not have a clear border or frontier: we see things as changing gradually from having these properties to not having them; we see as true that, if $n$ is available, $n$ +1 cannot be, suddenly, unavailable.

The problem with degrees of truth is that the only logic with which we are confident requires the Bivalence Principle. The Fuzzy Sets Theory [Zadeh 1965] applies *fuzzy logic* to obtain a graduality in the belongingness of elements to a set, but it seems impossible to apply it to infinite sets.

So we are left with supervaluations: we have to consider a *penumbra zone* inside which numbers become gradually unavailable. In order not to have a bivalent contradiction, it has to be impossible to talk about the elements, numbers in our case, that are inside the penumbra: they must be excluded by the theory so that we can no longer apply induction to them (that would give a contradiction). Actually, in this solution it is also admitted that Bivalence fails, only that the failure is assumed to happen inside a zone, the penumbra zone, which is excluded from the theory: the theory is no longer applicable inside the penumbra.

Thus, if we exclude a frontier number, we have to admit that, if there is a property that changes gradually, then Bivalence fails (although this failure can be confined to a penumbra zone). (II') must be partly false and should be substituted by:

(II' b)  a man with $n$ +1 \$ is 1/10[ix] less poor than a man with $n$ \$

which no longer authorizes to conclude (III').

Is "infinity" also a vague concept? Saying that "you can add '+1' an infinite number of times without ever obtaining an infinite number" seems parallel to saying that "you can give a poor man '+1 \$' a billion times without his ever becoming a billionaire": the latter reasoning is false, so how could the former be true?

There cannot be a "frontier" natural number because it would violate the theorem which says that "finite $n$ implies finite $n$+1", but the three dots "..." that we inevitably have at the end of any infinite series seem to conceal a penumbra zone inside which numbers can become gradually infinite. Inside this zone, at least, (II) could be substituted by:

(II b)  $n$ +1 is $1/\infty$ less finite than $n$

"$1/\infty$", the inverse of infinity, has no sense in ZFC, but it does in other systems.

In the next section we are going to see different arguments that suggest that, even in ZFC, what is valid for the "first" numbers may not be valid for all of them: if only the "first" numbers can properly be said to be finite, then there should be a gradual change from finite to infinite. How does ZFC cope with graduality?

## 7. The bivalent solution in ZFC

In the usual von Neumann ordinals, to obtain all the successors of 0, the void set, the successor operation has to be repeated an infinite number of times. Do the numbers become gradually infinite through this infinite repetition? They cannot become infinite in set theories that apply Bivalence and cannot treat graduality: numbers have to be either finite or not finite –i.e. infinite (notice that, then, a number is *called* "infinite" just because it is not completely finite). Its solution is to introduce *limit ordinals*: ordinals, other than 0, that are not the successor of any other ordinal, but that appear after any infinite series of them ending in a suspension points sign "...".

A limit, as a solution to vagueness, is a mix between the epistemic solution (as it is a sharp –not vague– element, $\lambda$, from which we can continue the series: $\lambda$, $\lambda$+1, $\lambda$+2, ...) and the supervaluations solution, as its predecessors have to end within a vague zone (which we represent by the "..."). This three dots sign acts like a newly added sign, because it is not included in the language of formal logic (only in its meta language), and its meaning should therefore, as is expected with "$\in$", be regulated by the axioms (there are reasons to believe that its internal definition is not unique).

The first, or smaller, limit ordinal, "$\omega$", is usually defined in the equiconsistent NBG (von Neumann-Bernays-Gödel) version of set theory, as the set of all finite ordinals {ordinal $x$: finite $x$}, so all its elements are finite by definition. Formally, $\omega$ works like a constant and is usually identified with the class "$N$" of the natural numbers; therefore, by definition: $\omega$ =

$N$ . But is $N$, so defined, the "real" class of all natural numbers (understood as all the elements that comply with Peano axioms)? It might not be if there were infinite natural numbers, because all the elements of ω have to be finite by definition. One consequence is that the theory cannot have ordinals such as ω/2, as it would be an infinite predecessor of ω.

Actually, it is shown in Model Theory that Peano axioms must have models of any cardinality, and the non-denumerable ones would contain infinite natural numbers, as you cannot have a non-denumerable amount of (internally) finite numbers –you should be able to use those same numbers to count the amount. Subsequently, "$\omega = N$ " is but a particular model. (I will call "Peano number" any element of any system complying with Peano axioms, be it finite or infinite).

That ω contains an infinite number of elements that are, all of them, finite implies some consequences that seem paradoxical (*paradoxos*: unexpected, unbelievable –although not necessarily a formal contradiction) given the properties of ordinals:

i) If we take any of the *initial segments* (without the 0, in this case) of the first $n$ numbers, {1}, {1, 2}, ... ,{1, 2, 3, ..., $n$}, the "size", or number of elements, of *each* segment is equal to its largest member, $n$ ; and the number of segments, up to any particular one, is equal to the number of elements, and therefore to the largest element, of the largest segment: that seems to contradict the assertion that when we consider *all* the segments, there are an infinity of them without any of its members being infinite.

ii) *Each* element of ω, being finite, has many, many more successors (an infinite number) than predecessors (a finite number); therefore, almost *all* the elements of ω are behind it. Each element must belong, then, to some finite "first part" of ω.

iii) The Tristram Shandy Paradox: Tristram decided to put in writing the history of his life, but soon discovered that he needed $n^2$ years to write the history of his first $n$ years; subsequently, as in the year $n$ he would have written only the history of his first $\sqrt{n}$ years, the fraction of his life to be written, if he was to live $n$ years, would be $\sqrt{n}/n = 1/\sqrt{n}$ . So, if he was to live eternally, as lim $(n\to\infty)$ $1/\sqrt{n} = 0$ , Tristram could write only a negligible (infinitesimal) part of his life. But, as Bertrand Russell[viii] noticed, if Tristram was to live eternally, as *each* year $n$ would be written (in year $n^2$), then *all* the years of his life, and not only a negligible fraction of them, would be written.

The previous paradoxes seem to rely on the following: *each* number can be reached in a finite number of steps, but any finite number of steps can reach *all* the numbers. In other words, in order to reach *each* natural number, $n\in\omega$, only a finite number of "+1" are needed; in order to reach *all* the numbers in ω, an infinite number of "+1" are needed. We represent "for each" by "$\forall n$" and "all in ω" by "$\{x\in\omega: \varphi(x)\}$": although in formal logic *for each* and *for all* are synonymous, there seems not to be the same extension in set theory if we take *for all* as referring to the totality of elements in an infinite set. The above paradoxes suggest that what we say for *each* number is valid for *all* the first *available* numbers, but not necessarily valid for *all* numbers (including the last *non-available* ones). Let us see a theory about non-available numbers.

## 8. The finite analysis solution

In his article [Mycielski 1986], **Jan Mycielski** demonstrates that any theory (a set of sentences) *T* has a counterpart *Fin (T)* with finitistic characteristics. The basis of the equivalence between *T* and *Fin (T)* is a metatheorem:

**Mycielski Theorem** ([Mycielski 1986], this version is from [Lavine 1994, p. 273]):

"If Φ is a sentence in the language of *T* and Φ' is a regular relativization of Φ, then:

*Φ is a theorem of T if and only if Φ' is a theorem of Fin ( T )* "

A formula Φ' of *Fin (T)* is said to be a "regular relativization" of a formula Φ of *T* if Φ' is obtained from Φ bounding all its quantifiers –that is, changing $(\forall x)$ and $(\exists x)$ to $(\forall x\in \Omega_p)$ and $(\exists x\in \Omega_p)$ for all variables *x*– in such a way that whenever $\Omega_q$ occurs within the scope of a quantifier bounded by $\Omega_p$ in Φ', then $p < q$ (p and q rational numbers to allow intercalation of indexes). (Notice that, in spite of the fact that the *Ω* s are intended to be finite sets, they could fail to be (Dedekind) finite if their indefiniteness was to allow a self-pairing of their elements).

Informally speaking, the Mycielski Theorem appears, if applied to ZFC, as a stronger version of Skolem's paradox: all that can be said, or established, with transfinite ordinals (including the inaccessible cardinals) can be said by talking not only about countable sets, but also about finite ones. Like Skolem's Paradox, the Theorem does not imply any contradiction, formally speaking, but it implies that a theory that we expected to be about infinite entities can be, as well, reinterpreted as true about finite entities.

In his book [Lavine 1994], **Shaughan Lavine** shows that *Fin (T)* can be interpreted as a theory of indefinitely large, but finite, quantities that could be described in a meta-language that did not involve infinite sets, according to his vision of infinity as an extrapolation of the concept "too many to count" (I follow Lavine's version).

It is a consequence from the Mycielski Theorem that *T* is consistent if and only if *Fin (T)* is consistent: the only significant difference between *T* and *Fin (T)* is that the latter has models in which the *Ω* s demarcate domains that are indefinitely large but finite [Lavine 1994, pp. 274-5].

As we add those new unary predicate symbols, $\Omega_p$, to *T*, we will need some additional axioms to regulate its use, and define them implicitly. Let us first see a motivation of the axioms through an example, drawn from arithmetic, presented by Lavine [1994, pp. 254-6], that also introduces his concept of *indefinitely large*:

"We learned that, say, 2 + 2 = 4, in part by taking a pile of two beans and another pile of two beans (...) Where did the beans come from? They come from some indefinitely large supply –say, a bucket of '$\omega_0$' beans.

"(...) The simplest arithmetical process that lets us use up beans is adding one –applying the successor operation. The fact that our pile of $\omega_0$ beans is exhaustible is expressible by the fact that the following sentence, written using bounded quantifiers, is false:

(1) $(\forall x< \omega_0)(\exists y< \omega_0)\ y = x+1.$

"Note, however, that the following sentence, which reflects our actual practice of getting more beans when we run out, is true:

(2) $(\forall x< \omega_0)(\exists y< \omega_1)\ y = x+1.$

"We could have started with $\omega_1$ beans, instead of $\omega_0$ , and used up all $\omega_1$, but then we would simply have had to get $\omega_2$ beans:

(3) $(\forall x< \omega_1)(\exists y< \omega_2)\ y = x+1.$

"We seem to be forced to an infinite series of statements like (2) and (3), but in fact we are not. Statements like (2) and (3) are more than superficially similar. We know that $\omega_0$ is an indefinitely large number with respect to our present interests, but nothing tells us how large –if we could place an upper bound on the size of w$\omega_0$, that would show that $\omega_0$ was not indefinitely large in the relevant sense. Thus, $\omega_1$ could perfectly well have played the role of $\omega_0$. Analogously, $\omega_2$ could perfectly well have played the role of $\omega_1$. The $\omega$s are, in a sense, indiscernible. When we codify that intuition about the indefinitely large, (3) will turn out to be a logical consequence of (2).

"From now on we switch almost completely from indefinitely large numbers, $\omega$s, to indefinitely large predicates, $\Omega$ s."

An interpretation of these new concepts is offered in [Lavine 1994, pp. 260-2]: "Indefinitely large is a notion that depends on some idea of *availability* of mathematical objects –for example, through description in a canonical notation– that could be taken to be primitive. Availability will not be part of our official version of finite mathematics, but it is worth giving a semiformal account of it nonetheless. (...) Fix a (finite) set $A$ of available objects and a (finite) set $F$ of availability functions, possibly many valued. A set $S$ *is indefinitely large* (over $A$ and $F$) if $A \subseteq S$, that is, if everything that is available is in the set.

"(...) A set $T$ is *indefinitely large with respect to* a set $S$ if everything that is available in virtue of the availability of the members of $S$ is in $T$. (...) If $\Omega_0$ is the first indefinitely large set under consideration and $\Omega_1$ the second, then $\Omega_0$ is indefinitely large with respect to what is actually available, while $\Omega_1$ is indefinitely large with respect to what is actually available and also with respect to what would be available in virtue of the availability of what is in $\Omega_0$.

"(...) It is a mistake to think of all of the members of $\Omega_0$ as being available. All that follows from our conception of $\Omega_0$ as indefinitely large is that if something is available, then it is in $\Omega_0$, not the converse."

You should notice, when reading the next section, the similarity between *indefinitely large* and *non-standard*, both concepts trying to grasp the epistemic fact of the existence of *non-available* natural numbers and, in general, of elements in the theories impossible to define. Such existence is established not only as something found in practice, but also as theoretically unavoidable.

Four axioms are added to regulate the use of these $\Omega$ s [ix]. The first three axioms[x] simply regulate that the extension of the $\Omega$ s is big enough to contain all the theory; the fourth axiom regulates that the $\Omega$ s are indiscernible:

Axiom (**4**): for p < q , p < r , q and r less than s for all s such that $\Omega_s$ appears in $\Phi$, the formula $\Phi$ regular, and the free variables in $\Phi$ among $x, x_1, \ldots, x_n$ ,

$$(\forall x, x_1, \ldots, x_n \in \Omega_p)( (\forall x \in \Omega_q)\Phi \leftrightarrow (\forall x \in \Omega_r)\Phi ).$$

axiom (4) formalizes the intuitive idea that all that matters about $\Omega_q$ is that it is indefinitely large. If we could not replace $\Omega_q$ with any given larger $\Omega_r$, that would mean that there was some upper bound to the possibilities for $\Omega_q$, contrary to the idea that it is indefinitely large.

Let us now consider ***Fin*** **(ZFC)**: it will result from adding the four axioms to the regular relativization of the axioms of ZFC. The relativized version that interests us is that of the Axiom of Infinity, " $\exists x$ ({ }$\in x \wedge \forall y$ ($y \in x \rightarrow y' \in x$) ", which relativized version Lavine [1984, p. 284] calls "of a Zillion":

**Axiom of a Zillion:**

**(Z)** $(\exists x \in \Omega_0)$ ({ }$\in x \wedge (\forall y \in \Omega_1)$ $(y \in x \rightarrow y' \in x)$ )

Says that there is an $x$ in $\Omega_0$ such that 0 belongs to $x$ and, if 0 is in it, so is 1, and so forth (0 should be in any $\Omega_i$ by the relativized version of other axioms).

Lavine concludes (underlining added):

"Thus, either $x$ will be infinite, a familiar possibility we can afford to ignore here, since Fin (ZFC) is to be a theory of hereditarily finite pure sets, or –to break the chain– there will be some number in $\Omega_1$ and in $x$ whose predecessor is in $x$ but not in $\Omega_1$. (...) Zillion declares that there are gaps in the set of available numbers (...) there is a number that is available that is so large that not all of its predecessors are available to any degree we actually employ. (...) The number

$$10^{10^{10^{10}}}$$

is my standard candidate (...) there must be a number less than it that is available without its predecessor being available to any actual degree, and that is a number of the sort being postulated by the axiom. Zillion says that there is a number available to us that is not available from below (in the precise sense already spelled out that its predecessor is unavailable, and hence we have not counted up to it)."

But, a "natural number" that comes after a gap, with non-available predecessors, is surely not (Dedekind) infinite (and therefore not a natural number as defined in set theory)?

## 9. Self-pairing of indefinitely large numbers

If $l$ is an instance of those natural numbers (seen as finite ordinals) with non available predecessors postulated by the Axiom of a Zillion, it appears that such a $l$ can be effectively self-paired: if a natural number, $n$, is "available" (not indefinitely large), so should be $n$ +1 (because the "successor operation" is available) and so should be the repetition of

"+1" an available number of times, subsequently, $2n = n +1+ \dots_{n \text{ times}} +1$ must be available, for any available $n$, and all of them will be then smaller than $l$ ; that gives us a pairing, $n \rightarrow 2n$, between all the available natural numbers without any non-available predecessor and all the available pair numbers without non-available predecessors. As $l$ contains a non-available predecessor, it must contain all the available natural numbers before any non-available, and then the self-pairing $n \rightarrow 2n$ is valid for the subset of the "first" (available without any non-available predecessor) elements of $l$ ; actually, that self-pairing is the same as that of the even numbers *into* (purportedly *onto*) the naturals (without the 0 to simplify):

$$\begin{array}{cccc} 1 & 2 & 3 & \\ \updownarrow & \updownarrow & \updownarrow & \cdots \\ 2 & 4 & 6 & \end{array}$$

just that in the latter case the three dots "..." are intended to represent an infinite number of elements, while in the former case they represent only an indefinitely large number of elements. But "if $n \in l$, then $2n \in l$" is equally true (alternatively, equally false) for all the available numbers as "if $n \in \omega$, then $2n \in \omega$" is true for all the natural numbers; subsequently, the basis for considering that there is a pairing *onto* is the same in both cases.

For the remaining elements (non-available, or available but with non-available predecessors) we can simply consider the identity pairing $\nu \rightarrow \nu$ ; which gives us for the predecessors of $l$:

$$\begin{array}{cccccc} l & l-1 & l-2 & & l/2 & \\ \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow & \cdots \\ l & l-1 & l-2 & & l/2 & \end{array}$$

Composing both self-pairings, we will have a self-pairing for all the elements of $l$ , $\{1, 2, 3, \dots ; \dots , l/2, \dots , l-1, l\}$ with its proper subset $\{2, 4, 6, \dots ; \dots , l/2, \dots , l-1, l\}$ (containing only the pair available without non-available predecessors in addition to all the non-available and all the available with available predecessors):

$$\begin{array}{ccccccccc} 1 & 2 & 3 & & l/2 & & l-1 & l \\ \updownarrow & \updownarrow & \updownarrow & \dots ; \dots & \updownarrow & \cdots & \updownarrow & \updownarrow \\ 2 & 4 & 6 & & l/2 & & l-1 & l \end{array}$$

the "... ; ..." signs expressing that we have the composition of two pairings, and also showing that there is a "gap" (of availability) between the two series. But we "know" that the series 1, 2, 3, ... , $l/2$, ... , $l-1$, $l$ is really continuous and that what is valid for the available elements must be also valid for the non-available in between, therefore we cannot but see the series of bijected pairs as also continuous:

$$\begin{array}{cccccccc} 1 & 2 & 3 & & l/2 & & l-1 & l \\ \updownarrow & \updownarrow & \updownarrow & \cdots & \updownarrow & \cdots & \updownarrow & \updownarrow \\ 2 & 4 & 6 & & l/2 & & l-1 & l \end{array}$$

$l$ is therefore (Dedekind) infinite

It may be argued that our reasoning is valid only in the meta-theory, because we are using the distinction *available / non-available*, which is not formally defined inside the theory, and "if $n$ is an available natural number so is $2n$" must be decided in the meta-theory, while "if $n$ is a finite ordinal so is $2n$" can be formally established using induction. Anyway, the decision to consider that there is a pairing *onto*, and not only *into* as required by common sense, is in both cases (for $l$ and for $\omega$) a decision in the meta-theory and, for it, a meta-distinction should be as good as an internal one.

In any case, such self-pairing of $l$ provides a meta **proof of the Euclidean interpretation** (which is a meta conclusion): we have a self-pairing of a set that we *know* has more elements than the corresponding subset, the set of its even numbers (such $l$ being a hereditarily finite ordinal –the elements of *Fin* (ZFC) are hereditarily finite pure sets); the fact that many numbers before $l$ are not available cannot change the fact that there are twice as many natural numbers as there are even numbers before $l$.

In spite of the fact that $l$ is intended to be a "natural number", it can be so only in a *traditional* sense, not according to the set theoretical definition, because $l$ cannot formally be said to be a (Dedekind) finite ordinal. A natural number in the "*traditional*" sense is a successor of 0 that can be, theoretically, attained by adding "+1" many continuously repeated times. $l$ can be obtained by repeating "+1", but an indefinitely large number of times and that is not a (Dedekind) finite number of times, there is no contradiction: $l$ is a natural number in the traditional (meta-theoretical) sense, but an infinite ordinal in the formal sense (inside the theory). $l$ must be an infinite Peano number.

## 10. Infinite -v- unavailable

From an ontological point of view, we have a clear intuition about the (Dedekind) infinite (traditional) natural numbers that appear in *Fin* (ZFC) (as indefinitely large, or available numbers with non available predecessors): they are natural numbers too large to count up to them, the existence of such numbers is obvious for us. Do their duals (the corresponding transfinite ordinals) in ZFC have an independent existence, or are they only a simplified version of their indefinitely large counterpart?

In other words, are ZFC and *Fin* (ZFC) two dual theories referring to two distinct mathematical realities, or is ZFC just a simplification of *Fin* (ZFC), both trying to approach the same matter (as happens with Newtonian and Einsteinian approaches to gravitation)? More precisely, is ZFC talking about the same reality with just a different, more simplified, language than *Fin* (ZFC)?

Let us see a *collage* of what Lavine has to say on the matter [1994, pp. 257-8, 265 and 252, including Note 7]: "Statements (2), (3), and (4) [reproduced above], which are all equivalent, are our counterpart for the statement

(5) $$(\forall x)(\exists y)\ y = x + 1$$

of ordinary arithmetic. Statement (5) (together with other parts of elementary arithmetic) guarantees that there are infinitely many numbers, and so it goes beyond finite arithmetic. It is obtained from statement (2) < $(\forall x< \omega_0)(\exists y< \omega_1)\ y = x + 1$ > by the process I have been calling extrapolation. Syntactically, the process is trivial –one just drops the bounds $\omega_0$ and $\omega_1$.

(...)

"The extrapolated theory is not a departure from the un-extrapolated one, a genuinely new theory, but merely a version of the un-extrapolated one in which the additional restriction is imposed that certain dependency –that on the bounds– takes its least complicated form. Nonetheless, setting the bounds equal is equivalent to making the extrapolation that we can start with so many beans that we can never need more. Of course that is a heap of beans that is no longer indefinitely large –it is infinitely large. The *theory* is not much changed by the extrapolation, but the classes of its *models* is changed in an important way: the class of the models of the extrapolated theory excludes finite models.

(...)

"Why shouldn't we take the attitude that, while it was once justified to endorse ordinary set theory, it is no longer justified to take it to be anything more than a convenient abbreviation in which the $\Omega$ s are suppressed, now that we see what was lurking behind the reasons we apparently had for accepting it? (...) In our present state of knowledge, the $\Omega$ s do not play any mathematical role. It would therefore, for methodological reasons, be a mistake to require them (...) the success of a method of extrapolation entails the apparently stronger possibility of an elimination.

"Though it is not my motive for developing the finite mathematics of indefinitely large size, the possibility of developing it shows that one could coherently deny the existence of infinite sets without doing violence to mathematical practice. (...) Though the existence of a basis for extrapolation does not force us to deny the existence of infinite sets, it does undercut with what are usually thought to be the chief positive arguments for their existence."

Is there any reality under these extrapolated infinities? The (meta) self-pairing of an indefinitely large number, that I have shown, seems to prove that we do not need anything different to have a (Dedekind) infinity. B. Russell would require us to apply "Ockham's Razor": we should not introduce "really" infinite sets (we expect them to be much larger than any natural number, no matter how large) in our interpretation of the extrapolated theory, unless some good reason force us to do so. The traditional relation between Mathematics and Physics does not favor at all such an interpretation: one of the most notable achievements of Physics in the 20th Century was the measurement of the size and age of our Universe (which is subsequently finite) and of Planck's length and time, which seem to be minimums for physical entities. When an infinity appears in any physical theory, it is undoubtedly assumed that such theory has to be "renormalized" in order to suppress such infinity.

During the same period, mathematicians developed a set theory with not only (purported) actual infinities, but with an endless increasing marvelous structure of infinities, each one incomparably larger than its predecessors; should we now reinterpret them as a simple extrapolation of indefinitely large (humble) natural numbers? The beauty of a formal structure does not depend on its interpretation: in *Fin* (ZFC), the condition "$\in \Omega_i$" implies that the duals of the ordinals of ZFC are all included in a subset of $N$, and the infiniteness of $l$ implies that the duals of the natural numbers are all included in $l$. But ZFC, even expanded with large cardinals axioms, does not change when reinterpreting:

$$\omega = l \quad \text{and} \quad \Omega = N$$

that is, that the (traditional) natural numbers (those obtainable through operations over the previous ones) do attain all the possible ordinal numbers (including the largest inaccessible cardinals) although there are gaps of availability in the series (the first gap coming after the numbers available before any non-available that we interpret as the finite numbers), those gaps having the same structure as the transfinite gaps in the classic interpretation, where:

$\omega = N$ and $\Omega$ = the class of all the ordinals (natural numbers + transfinite).

Actually, Tarski's Upward and Downward Löwenheim-Skolem theorems allow us to deduce the existence of a model for each one of these interpretations if we have a model for the other (I am assuming a generalized version of Compactness saying that "There is a model for a proper class if there is a model for any of its subclasses of transfinite cardinality", which must be true because, if not, there would be a model of maximal cardinality, against Upward).

Recall that the reasoning about size of sets, about whether or not they are "really" infinite (*l* is not), does not make sense from the formalist's point of view: formally we can only talk about equicardinality, not about sameness of size, because we do not have this concept in the theory; subsequently, formalism is indifferent about whether a set admitting a self-pairing is interpreted as having the same size as its proper subset or as being larger than any proper subset (in spite of being bijected with it).

Anyway, Lavine's reasoning about the existence of a number with a non-available predecessor is also an interpretation in the meta-theory, because the theory does not include the term *available*, so to have a better understanding we have to look at a theory that tries to formalize the vague concept of *availability* (and also including the inverse of infinity).

## 11. Non-standard infinitely large natural numbers

In a paper published in 1934[xi], Toralf Skolem showed how to construct a proper extension of the system of natural numbers that contained infinitely large natural numbers. These *Non-standard models of Arithmetic* were considered by mathematicians to be nothing more than a logical curiosity, until **Abraham Robinson**, in 1960, had the inspiration to use the inverses of those infinitely large numbers as infinitesimals (something considered impossible with the standard infinite ordinals), giving a sound foundation to the much used Differential Calculus. While those infinitesimals are studied by "Non-standard Analysis", their inverses, the infinite numbers, should be studied by "Non-standard Arithmetic" (both jointly constituting the "NSA" theory), but while NS Analysis has received a considerable amount of work, NS Arithmetic has had no development. The reason, as commented, is that those infinite numbers seems to be incompatible with the well established Cantorian transfinite numbers. Let us see why.

In [Robinson 1974], where the Compactness meta-theorem (significantly called "Finiteness Principle") is demonstrated also for higher order theories [pp. 26-7], it is shown [chapter II] that there must be a higher order non-standard model of Arithmetic, "*$*N$* " (anyway, to conclude the debate about the existence of non-standard models with higher order languages, let me refer to [Jané 1993], where strong arguments are produced to conclude [p. 85]: "First order logic is the right underlying logic for set theory. It offers us the means to extract all consequences from the axioms and presupposes nothing about sets").

$*N$ is said to have properties like the following [Robinson 1974, 3.1]:

"(ii) Every mathematical statement that is meaningful and true for the system of Natural Numbers is meaningful and true also for $*N$ : provided that we interpret any reference (...) in terms of a certain subset, called the set of *internal* entities of that type <the

distinction is essential because not all types of sentences are *absolute*, as shown in [Bagaría and Friedman 2001][xii]>. (...) However, all individuals of **N* are internal.

(...)

"(iv)**N* properly contains the system of natural numbers, *N* ; there is an individual in **N* which, according to the relation of order defined in *N* and **N* , is greater than all numbers of *N* ."

Are those infinitely large numbers (Dedekind) infinite? Being greater than all ordinary natural numbers, it seemed obvious to Robinson to call them "infinite" (underlining added) [after 3.1.4]:

**"**From now on, we shall call all natural numbers that belongs to *N* *finite*, while all other natural numbers <those belonging to **N#N* –the class of the elements of **N* not belonging to *N*> will be called *infinite.* (...) The finite natural numbers are the *standard* natural numbers according to a general classification introduced previously."

That "classification introduced previously" must be the one in [Robinson 1974, 2.11, third paragraph], where an element is said to be *standard* if, and only if, it belongs already to the original (not expanded) model, in which case it will always be *internal* (will belong to the expanded model); subsequently, an element will be *non-standard* if it does not belong to the original model, being then *internal* (belonging to the expanded theory) if it belongs to the expanded model, and *external* (the theory cannot talk about it) if it does not belong to the expanded model either.

Subsequently, here "*infinite"* needs not be "(Dedekind) infinite", as Robinson does not demonstrate that the non-standard can be self-paired. In order to avoid confusions, let us call a *number* "**(Robinson) infinite**" if it is larger (in an Arithmetical sense) than any standard natural number.

If we consider Arithmetic as defined inside ZFC, then it makes sense to define a self-pairing, and one can be actually shown, in the meta-theory at least, for any (Robinson) infinite number, $\nu$, just using a (composed) pairing like:

$$g(n) = 2n \qquad \text{if } n\in\nu \text{ is (Robinson) finite } (n\in N)$$

$$g(m) = m \qquad \text{if } m\in\nu \text{ is (Robinson) infinite } (m\in {}^*N\ \ N)$$

The zone between the (Robinson) finite and infinite numbers behaves as a *penumbra zone* –there is no frontier number– that hides the fact that the first part of the injection, $n\rightarrow 2n$, is not onto. We are forced again to the Euclidean interpretation.

May those infinite numbers be assimilated to the transfinite ordinals in ZFC? This seems problematic because the latter are the successors of $\omega$ that Robinson considers to be an element "external" to his theory, as $\omega$ is the smallest transfinite and it is said in [Robinson 1974, p. 51]:

"*There is no smallest infinite number*. For if $a$ is infinite, then $a \neq 0$, hence $a = b + 1$ (the corresponding fact being true in *N* ). But $b$ cannot be finite, for then $a$ would be finite. Hence, there exists an infinite number which is smaller than $a$ ."

If Robinson is right, this is a reason why NSA is a theory disjoint to ZFC: its infinite numbers, complying with all the properties of the usual ones (Leibniz's Principle), including that of always having a predecessor, cannot be the same as ZFC's transfinite numbers, which include limit ordinals that do not have a predecessor. We could, maybe, solve the problem by defining predecessors for the limit ordinals, just as we define a predecessor for 0, namely –1 (redefining also the ordinal operations to be commutative). That is, we could consider that the system of the ordinals can be adequately expanded in order to appear as a continuous series of elements as in NSA.

There is another reason why NSA and ZFC appear to be disjoint: the structure of the (Robinson) infinite numbers seems to be essentially different from the structure of the transfinite ordinals. Robinson uses an equivalence relation to show that his infinite numbers are structured in "*intervals*", more usually called (as in [Manzano 1989, pp. 219-21]) "**Z**-chains", because of their similarity with the class $Z$ of the integers ( ... , -2, -1, 0, 1, 2, ... ) and also called "*galaxies*" because the natural numbers can be assimilated to points in a topological (discontinuous) space and Robinson calls the set of points at a finite distance from a given point the "*galaxy*" of that point, which is equivalent to an *interval* (see [Pin 1990, pp. 234-7] for a formal extension of the concept). In [Robinson 1974, pp. 51-2] we are told:

"We define a binary relation ~ for the numbers of $*N$ by the condition that $a \sim b$ if and only if $|a - b|$ <absolute value> is finite (...) the equivalence classes determined by this relation are *intervals* of the ordered set of numbers of $*N$ . (...) Now let $a$ be an infinite natural number and let $D_a$ be the equivalence class of $a$ with respect to ~. Then the numbers $a -1, a -2, \dots , a -n, \dots$ , $n$ a number in $N$ , all exist since $a$ is infinite, and the numbers $a, a \pm 1, a \pm 2, \dots , a \pm n$ , $n$ a number in $N$ , all belong to $D_a$ since $|(a \pm n) - a| = n$ is finite. Moreover, the numbers just mentioned constitute the entire class $D_a$ ."

Such *intervals* $(\dots , a -2, a -1, a , a +1, a +2, \dots)$ seem similar to any of the enumerable series in which the transfinite ordinals ($\lambda , \lambda +1, \lambda +2, \dots$ ; $\lambda$ any limit ordinal) are distributed, although expanded in the descending sense, as $Z$ expands $N$ ; nevertheless, the structure of the *intervals* should better be compared with that of $Q$, because they form a dense order: in between any two *intervals, a, b,* you can have infinitely many others, as can be easily seen (e.g., $(a+b)/2 \pm n$).

What cardinality does the class containing all these *intervals* in $*N$ have? By Tarski's Upward Löwenheim-Skolem model-theoretic theorem we know that the non-standard model $*N$ may be taken as having any transfinite cardinality, $\aleph_\alpha$ , with possibly $\alpha > 0$, and, as any *interval* has cardinal $\aleph_0$ , there must be $\aleph_\alpha$ intervals. Actually, as the cardinality of the model has no limit, it has to be a model with the cardinality of a proper class, the same as $\Omega$, the class of all ordinals.

Considering the two previous paragraphs, may those *intervals* be considered as an expansion of the transfinite ordinals, adding predecessors to the limit ordinals in order to fill (densely) the gaps before them? The usual interpretation, that the two systems are disjoint, seems like saying that $N$ and $Q$ are two disjoint systems because in $N$ we do not have, in many cases, inverses: $N$ and $Q$ are not disjoint, but one of them an extension of the other, could it not be the same case between $*N$ and $\Omega$ ? It seems more reasonable than having two disjoint infinite number systems, but it requires the previous reinterpretation: $\Omega = N$ (in the explained sense).

## 12. Infinitesimals

As said, Non-standard Analysis has received much more attention than Non–standard Arithmetic, the reason being its capacity to introduce formally elements that can be assimilated with the infinitesimals introduced, mainly by Newton and Leibniz, in Calculus and that, in spite of their capacity to solve many problems, were substituted by the epsilon–delta limits because they lacked a sound foundation. Robinson introduced the infinitesimals and related concepts through the following definitions [1974, pp. 55-7]:

"Let $R$ be a higher order non-standard model of Analysis (...) From now on we shall refer to all individuals of $R$ as *real numbers*, reserving the name *standard real numbers* to the individuals of $R$. In order to simplify our notation we shall write $a\in {}^*R$ (or $a\in R$) to indicate that $a$ is a real number (a standard real number) (...)

"$^*R$ is non-Archimedean since it contains numbers that are greater than all numbers of $R$, which constitutes a subfield of $^*R$. Thus, for some number $a$ in $^*R$,

$$1< a,\ 1+1< a,\ ...,\ 1+1+...+1< a,...,\qquad \text{and so on}$$

where we add 1 any (finite) number of times to itself (...)

"A real number $a\in {}^*R$ will be called *finite* if there exists a standard number $m\in R$ such that $|a| \leq m$, while any other number of $^*R$ will be called *infinite*. (...) A number $a\in {}^*R$ will be called *infinitesimal* or *infinitely small* if $|a| < m$ for *all* positive numbers $m$ in $R$. By this definition 0 is infinitesimal. (...) A number $r\in {}^*R$, $r \neq 0$, is infinitesimal if and only if $r^{-1}$ is infinite. If $a - b$ is infinitesimal, then we say that $b$ is *infinitely close* to $a$ (...) For any finite real number $a$ in $^*R$ we call the uniquely determined standard real number which is infinitely close to $a$ the *standard part* of $a$ (...) we call the set of real numbers which are infinitely close to $a$ the *monad* of $a$ ."

If $r$ is a standard real and $\varepsilon$ an infinitesimal, then $r+\varepsilon$ is a non-standard real infinitely close to $r$, the standard and the non-standard are intercalated in $^*R$; in order to see how they are ordered by size, let us consider a subclass of $^*R$ that constitutes the "skeleton" of its positive elements, the class of the natural numbers plus their inverses, including the non-standard (the inverse of a non-standard is also non-standard):

$$\{0, 1, 2, 1/2, 3, 1/3,..., \nu, 1/\nu, \nu +1, 1/\nu +1,..., 2\nu -1, 1/2\nu -1, 2\nu, 1/2\nu,...\}$$

where $\nu$ is a generic non-standard number; $2\nu - \nu$ is infinite, so $\nu$ and $2\nu$ belong to different intervals; 0, 1, 2, 3, ... belong to the same interval, usually called the *principal galaxy*. If we order the elements by increasing size, we have:

$$0;...,1/2\nu,1/2\nu -1,...;....,1/2\nu +1,1/2\nu,...;...,1/3,\ 1/2,\ 1,\ 2,\ 3,...;...,\ \nu,\ \nu +1,...;...,\ 2\nu -1,\ 2\nu,...$$

I use the "...;..." sign to show that there is a "gap" between the two series: there is no successor of 0, there is no frontier number between, for example, the series, $\nu$, $\nu +1$,..., and ,...,$2\nu -1$, $2\nu$. We have, then, the following categories, ordered by size (calling *observable* the finite and non infinitesimal, and considering that if a number is standard so is its inverse):

zero (standard) – infinitesimal-non zero (non-st.) – observable (st.) – infinite (non-st.)

for what we have seen about intervals, there is in this class a denumerable amount of observable (and therefore of standard) numbers, but whatever cardinality of infinite (and therefore of non-standard) numbers, as well as of their inverses (also non-standard), the infinitesimals.

Inside $*R$ there is no such clear separation between standard and non-standard because we have any "hyperreal" number (a name that provides a better distinction than Robinson's for the elements of $*R$) of the form $r+\varepsilon$, where $r$ is any standard real and $e$ any –possibly negative– infinitesimal (non-standard, then, as will be $r+\varepsilon$). There is no standard real between $r$ and $r+\varepsilon$, or between 0 and $\varepsilon$, a proof that the continuum (e.g. the real line) is more dense than the standard (classical, recall) real numbers: the hyperreals provide a more dense division of the real line; the higher the cardinality of $*R$ (that of the model), the denser; actually, as there is no limit to the cardinality of $*R$ (other than that, potential, of the proper classes), the hyperreals provide the ultimate division of the continuum.

## 13. Finite non-standard numbers

There is an alternative solution that allows to have non-standard and transfinite numbers as members of the same system: to consider that the non-standard numbers are not "really" infinite, as Robinson "calls" them, but, all of them, (Dedekind) finite and, subsequently, predecessors of the transfinite; this is the solution proposed by the following version of non standard analysis.

A. Robinson relied on model theory to develop his theory of Non-standard Analysis, but suggested alternative methods, like ultraproducts, and also other methods less usual in model theory [Robinson 1974, p.47]: "Differing from our approach more radically, one might use axiomatic set theory rather than type theory for the development of higher order Non-standard Analysis."

Following this suggestion, **Edward Nelson** developed an axiomatic version of Non-standard Analysis that he called "Internal Set Theory" ("IST") [Nelson 1977]: "In addition to the usual undefined predicate $\in$ of set theory we adjoin a new undefined predicate *standard*. The axioms of IST are the usual axioms of ZFC plus three others, which we will state below.

"*All theorems of conventional mathematics remain valid.* (...) We chose to call certain sets standard, but the theorems of conventional mathematics apply to all sets, nonstandard as well as standard" [p. 1165]. Calling "(I)","(S)" and "(T)" the three new axioms (Idealization, Standardization and Transfer), Nelson's version is therefore: IST = ZFC + (I) + (S) + (T).

[Nelson 1977, Appendix] contains a proof of a meta-theorem, by William C. Powell, which says that IST is *conservative* relative to ZFC (any statement, meaningful in ZFC, that can be proved with IST can also be proved with ZFC alone). As IST is said not to add any new element but just to distinguish, within the preexisting, between the standard and the non-standard, Leibniz's Principle follows from the fact that the non-standard are not really "new".

To introduce the new axioms, we need some definitions and abridging:

- an "*internal*" (or "*classical*")[xiii] formula (sentence, property, etc.) is one meaningful in ZFC, that is, one not containing the new term *standard* (otherwise it is *external*).

- "$\forall^s x$ P" means $\forall x$ (St $x \rightarrow$ P), where "St" is the formal abridging of *standard*
- "$\exists^s x$ P" means $\exists x$ (St $x \wedge$ P)

**Transfer** axiom:

Let *F(x, A, B ..., L)* be an internal formula with all its parameters, *A,B,..., L* taking only standard values (the number of parameters itself must be standard otherwise the formula could not be internal), then:

**(T)** $\forall x\, F(x, A, B,..., L) \leftrightarrow \forall^S x\, F(x, A, B,..., L)$

That is, *F* holds for all *x* as soon as it is valid for all standard *x*[xiv].

Transfer has a dual form, obtained negating both terms:

**($T^{-1}$)** $\exists\, x$ with $G(x, A, B,..., L) \leftrightarrow \exists^s x$ with $G(A, B,..., L)$

In particular, if there is a *unique x* such that *G* (with all the conditions as in (T) ), then *this x* must be standard. That means that any concept well (uniquely) *defined* within ZFC is standard (in particular, ω is, although not used when working within IST). Subsequently, a non-standard cannot be (internally) defined. Can there be standard elements that cannot be defined? Transfer does not forbid them.

**Standardization** axiom

Let *P(x)* be any property (possibly *external* –containing the new term "standard")

**(S)** $\exists^s A$ such that $\forall^s x\, (x{\in}A \leftrightarrow x{\in}E \wedge P(x))$

The *Standardization* axiom is needed in order to regulate the use of the term "standard" in set forming: there must exist a standard subset containing all the standard elements, of some given set *E*, that comply with *P(x)*, although not necessarily containing all the non-standard that comply and, maybe, containing some non-standard that do not comply (the non-standard must remain vague to cope with the graduality). Applying (T): "two *standard* sets will be *equal* as soon as they have the same standard elements". It follows that the standard set postulated by (S) is unique.

**Idealization** axiom:

Let R($x$, $y$) be an internal binary relation and *F* whatsoever standard and finite set, then

**(I)** $\forall^s F$ (finite $F \rightarrow \exists x\ \ \forall y{\in}F\ \mathrm{R}(x, y)$ ) $\leftrightarrow \exists x\ \forall^s y\ \mathrm{R}(x, y)$

As Compactness, (I) allows to establish the existence of non-standard sets:

Taking $\mathrm{R}(x, y) = (x{\in}E \wedge (y{\in}E \rightarrow x \neq y))$, for any infinite set *E* we can find a convenient $x{\in}E$ for each finite subset *F* (a fortiori, for each finite and standard), then by (I):

Theorem (**1**):

$$\exists x \in E \; \forall^{s} y \; (y \in E \rightarrow x \neq y)$$

" *there is a non-standard* element *in any infinite* set"

this implies that non-standard sets exist (as there are infinite sets in ZFC). In particular, as there is an infinity of natural numbers, there are non-standard natural numbers.

Taking now $R(x, y) = (x \in N \wedge \text{finite } x \wedge (y \in N \rightarrow y < x))$, as there is such an $x$ for every finite and standard set $F$ (just take $n+1$, being $n$ the largest natural number in $F$ –if none take 0, there must be a largest $n$ because $F$ is finite), then by (I):

Theorem (**2**):

$$\exists x \; (x \in N \wedge \text{finite } x \wedge \forall^{s} y \; (y \in N \rightarrow y < x))$$

"there is a finite natural number $\nu \in N$ larger than all standard $y \in N$ "

Contrary to Robinson's NSA, in IST a non-standard natural number can be finite.

But (I) has also some important consequences that go beyond its intended use in establishing the existence of non-standard sets.

## 14. Consequences for infinity

Let us take as ZFC's Axiom of Infinity ("AI") the version which says that "ω is a set" (without this or an equivalent axiom $\omega = \{\text{ordinal } x\text{: finite } x\}$ could well be a proper class, not having successors):

**AI** $\quad$ *Set* ω $\quad$ (ω is a set)

Can AI be deduced from (I) ? Just take in (I)[xv], $R(x, y) = \mathit{Set}\ x \wedge (y \in x \leftrightarrow \text{ordinal } y \wedge \text{finite } y)$; "$\forall^{s} F$ (finite $F \rightarrow \exists x \; \forall^{s} y \in F \; (\mathit{Set}\ x \wedge (y \in x \leftrightarrow \text{ordinal } y \wedge \text{finite } y))$" is true because $F$, being finite, will contain only a finite number of finite ordinals and, letting $n$ be the largest of them (if any, take $x = 0$), we have that $x = n+1$ will comply, as *Set* $n+1$ (being a finite ordinal) and being a von Neumann ordinal just the set of all its ordinal predecessors, which are finite by construction in this case; then by (I):

$$\exists x \; \forall^{s} y \; (\mathit{Set}\ x \wedge (y \in x \leftrightarrow \text{ordinal } y \wedge \text{finite } y))$$

equivalent to $\quad \exists x \; (\mathit{Set}\ x \wedge \forall^{s} y \; (y \in x \leftrightarrow \text{ordinal } y \wedge \text{finite } y))$

and applying (T'), there must be a standard such $x$; that proves

Theorem (**3**):

$$\exists^{s} x \; (\mathit{Set}\ x \wedge \forall^{s} y \; (y \in x \leftrightarrow \text{ordinal } y \wedge \text{finite } y))$$

but such a standard set will have the same standard elements as ω and, as seen, in IST two standard sets having the same standard elements are equal, then (being ω standard because classically defined, and considering its definition), we have as corollary:

Theoreme **(4)**:

$$Set\ \omega$$

AI is redundant in IST.

But deducing AI from (I) + (T) has no interest unless these two axioms are more obviously true than AI. Are they? To start with, AI is far from obvious, on the contrary: if ω were not a set, but a proper class, then it could not be considered as a completed totality; it should be considered as only *potentially* infinite, in agreement with major philosophical opinion, including Aristotle, Aquinas, Leibniz and Kant [Soler, Soler and Soler, chapters I, V, VI and VII], while ω being a set implies that we can have all the natural numbers as a completed totality, an actual infinity; it is not surprising that Cantor (who anyway called the proper classes "absolutely infinite inconsistent multiplicities", suggesting that the sets are only "relatively infinite") had to face fierce opposition. Actually, "*Set* σ" implies that σ is not absolutely infinite, a "real" infinity.

(I) and (T), in contrast, cannot be disputed because they are the formalization inside the theory of theorems from other theories that are seen as true in the meta-theory of ZFC: (I) just says about the standard sets the same that Compactness says about its models, that what is true for the elements of every finite subset is also true for all the elements.

In order to see that (T) is also a version of a theorem, we need to consider first another theorem, derived from (I), relevant about infiniteness:

Taking in (I), R(*A, y*) = (*A* is finite and $y \in A$), as R is true for all the elements of each standard and finite *F* (just take *A* = *F* ), then by (I):

Theorem **(5)**[xvi]:

$$\exists x\ (\text{finite}\ x \wedge \forall^s y\ (y \in x)\ )$$

"There exists a finite set containing all the standard elements of IST"

As there would not be a classical definition, there is not an internal set containing just all the standard elements of the theory, but there will be finite sets containing all the standard in addition to other non-standard elements[xvii].

Returning now to (T), let us call "$A^s$ " the *relativization of a formula A to the standard sets*, the formula obtained replacing each occurrence of $\forall x$ by $\forall^s x$ and of $\exists x$ by $\exists^S x$; in [Nelson 1974, p. 1166] it is shown how: "By successive applications of (T) (working from outside in) we see that $A \leftrightarrow A^s$. Thus all theorems of conventional mathematics also hold when relativized to the standard sets. Conversely, to prove an internal theorem it suffices to prove its relativization to the standard sets."

Subsequently, we can take ZFC +(T) as a version of ZFC relativized to the standard sets. And then, if we take **F** as an instance of those finite sets that contain, according to

Theorem (4), all the standard, we will have for any sentence $\Phi$ of the theory (applying that "what is valid for all is valid for some"):

$$\forall x\, \Phi \rightarrow \forall x{\in}\,\mathbf{F}\, \Phi$$

and (applying the obvious principle that "what is valid for all the elements of a set will be valid for all the elements of any of its subsets", even if external –like the subset of all the standard elements of **F**):

$$\forall x{\in}\,\mathbf{F}\, \Phi \rightarrow \forall^{s} x\, \Phi$$

but, applying (T) $\quad \forall^{s} x\, \Phi \rightarrow \forall x\, \Phi$

then $\quad \forall x\, \Phi \leftrightarrow \forall x{\in}\,\mathbf{F}\, \Phi$

a parallel proof, applying ($T^{-1}$), shows that:

$$\exists x\, \Phi \leftrightarrow \exists x{\in}\,\mathbf{F}\, \Phi$$

(T) is, then, equivalent to saying that "any sentence of ZFC remains valid when relativized to an (externally) finite set containing all its standard elements", *a similar conclusion to that from Mycielski's Theorem*. Subsequently, (T) should be seen also as true because it can be considered as a version, inside the theory, of Mycielski's Theorem.

We may then take as the basic theory for non-standard analysis:

**IST– AI** = ZFC –AI +(I) +(S) +(T)

and also ( (S) being simply an extension of Comprehension to external, non classical, formulas) take <u>IST–AI as a substitute of ZFC</u>.

IST–AI may well be considered as a variant of *Fin* (ZFC) in which, instead of relativizing to the $\Omega_i$s, we relativize to the standard elements, something that must be equivalent, as the standard are all contained inside a finite set and we can tell (in the meta-theory) the same about the $\Omega_i$s. Both theories are talking about the same "reality": the natural numbers and the sets derived from them, distinguishing those that are available / standard from those that are not (although IST uses a solution –the gradual loss of availability inside the vague zones of non-standard– in the line of supervaluations, while *Fin* (ZFC), at least in Lavine's interpretation, tries to offer a solution in the line of that of ZFC, postulating the existence of a *limit* number that is available without its predecessor being available).

Let us now consider other relevant consequences for the infinity obtained from a particular cases version of Theorem (5): taking in (I), R($A$, $y$) = ($A$ is a finite part -of a set $E$- and contains $y$), as this is true for every standard and finite part of whatsoever $E$ (take $F$ = the unique standard subset defined by ($F \cap E$) ), then:

Theorem **(6)**:

"there exists a finite part $A$ of $E$ containing all standard elements of $E$ "

It does not imply that such a finite part contains "only" the standard elements of $E$, it will contain, in general, also non-standard elements. So it cannot be formally established, based on Theorem (6), that the subset of (only) the standard elements of a set is finite, but we see it as such in the meta-theory.

If *S* is a finite subset containing all the standard elements of an infinite set *I*, then $I \setminus S$ must be infinite and contain only non-standard elements of *I* (although not all of them). We have then the following powerful conclusion about infinity:

> "every infinite set is composed of a finite subset containing all its standard elements plus an infinite set containing only non-standard ones"

then, there are the non-standard elements that make a set infinite (here *infinite* is *Dedekind*).

Taking the most basic infinite set, $\omega$ , it will be composed of a finite (in the meta-theory) number of standard ordinals, that we can establish are finite, plus an infinite number of non-standard ordinals some of which we know are also finite – but are all of them finite? Let us first see how to use pairing, in IST, applying two consequences of the axioms and theorems about graphs and maps [Robert 1985, paragraphs 2.5.1 and 2.5.3]:

"A map $f{:}E \to F$ is standard precisely when the sets *E, F* and the graph Gr(*f* ) are standard sets.

"(...) Two standard maps $f,\ g{:}E \to F$ taking same values at all standard elements (of *E*) are equal."

Subsequently, in order to pair a standard infinite set with a proper standard subset (to be proper, some standard element must not be common) all we need to do is pair its standard elements (which, remember, will be contained in a finite set). IST tell us that we cannot, then, be sure that all the non-standard elements (in infinite number) are going to have their correlate in the pairing, providing an explanation for the Euclidean solution (recall that we do not have any explanation for the Cantorian) to Galileo's Paradox: **two infinite sets of different sizes can**, nonetheless, **be paired because inside the non-standard zone we cannot see that the pairing is really not onto** (is incomplete).

Can there be non-standard maps? According to ($T^{-1}$), if there is a non-standard map, it must be also a standard one. A Corollary is that a non-standard set cannot be mapped, because it should have a standard map and it cannot be such with a non-standard domain: a non-standard set cannot be said to be or not to be (Dedekind) infinite on the basis of self-pairing, because a pairing is impossible to show for them.

Subsequently, we cannot confirm, on the basis of self-pairing, that a non-standard natural number is (Dedekind) finite (actually, their standard elements could be self-paired –just taking the internal, or usual, self-pairing of the natural with the even numbers– although we cannot conclude from this that such non-standard is (Dedekind) infinite, because it should be standard for such conclusion).

Neither can we show that there is an infinite non-standard number on the basis that, if $\nu$ is any non-standard, the descending series, $\nu$, $\nu$-1, $\nu$-2, ... is unlimited, because, in spite of the fact that this series can apparently be self-paired, that would not be a standard pairing; actually, there must be some finite non-standard number by Theorem (2)[xviii].

It is this indetermination, about the finiteness of the non-standard, that makes the gradual change from finite to infinite possible and that solves the paradoxical properties that we have seen presented by the bivalent solution of limit ordinals. The infinite non-standard natural numbers must be external in order not to contradict (T), which establishes that if all the standard natural numbers are finite, so will be all the (internal –see note 14) non-standard.

Actually, in the usual presentations of IST all the non-standard natural numbers are considered classical finite ordinals (or are intended to be, as they can fail to be finite, as happened in *Fin* (ZFC)): Nelson departs from Robinson in calling "*unlimited*" the real numbers that are larger than any standard real number (that is, the non-standard real numbers –including the non-standard integers), presuming that all of them are finite and that, subsequently, they all possess the usual properties of real numbers –or of integers. An "*infinitesimal*" is, similarly to NSA, a real number smaller than any standard real, but in IST it is the inverse of some unlimited. In IST we will have, subsequently, the following categories of natural numbers and their inverses (the elements of {0, 1, 2, 1/2, 3, 1/3,..., ν –1, 1/ν –1, ν, 1/ν, ν +1, 1/ν +1,.... }) by increasing size:

zero (standard) – infinitesimal-non zero (non-st.) – observable (st.) – unlimited (non-st.)

quite similar to its structure in NSA, but in IST there are shown to be, in this subclass, only a finite number of observable, because standard, elements, and there seems to be only a denumerable number of infinite / infinitesimal (and therefore of non-standard) elements, because, 0, 1, 2,...;..., ν, ν+1,..., are all members of the classical *N*. As is said in [Robert 1985, p. 9]:

"We are *not going to add elements to the classical set N of natural numbers*, and we shall never refer to an 'extension **N* ' of *N* as Robinson initially did. But if *N* still represents the same classical set, it is also true that the new deduction principles –resulting from the new axioms– may give a psychological feeling of extension since they reveal elements that were unknown to ZFC."

It appears as if IST was a finitary dual version of NSA with a denumerable number of **Z**–chains of natural numbers, each chain being finite (although unlimited). But, even if *N* has not changed, do the new "revealed" elements imply that we now may "see" in *N* more elements than before? In other words, what cardinality has *N* in IST? We have seen in *Fin* (ZFC) how elements expected to be finite might be self-paired due to the vague zones introduced by the axioms about the indefinitely large $\Omega_i$s. Will the vague zones introduced by the standard / non-standard distinction have the same effect?

## 15. "Natural numbers" that are not a natural number

We need a pair of additional theorems,

Theorem (**7**):

$$\mathrm{St}\, E \wedge \mathrm{Fin}\, E \leftrightarrow \forall x\, (\, x \in E \rightarrow \mathrm{St}\, x)$$

"a set is standard and finite as soon as all its elements are standard"

Proof: if all its elements are standard, a set cannot be infinite because then it should contain, by theorem (1), some non-standard element; and if a standard and finite set, *E*, was to contain a non-standard element, ν, then:

$$\exists \nu \, (\nu \in E \wedge \forall^s y \, (\nu \neq y))$$

equivalent to $\exists \nu \, \forall^s y \, (\nu \in E \wedge \nu \neq y)$ because non-St $\nu$

and applying (I), in its "$\leftarrow$" sense:

$$\forall^s F \, (\text{finite } F \rightarrow \exists \nu \, \forall y \in F \, (\nu \in E \wedge \nu \neq y))$$

equivalent to $\forall^s F \, (\text{finite } F \rightarrow \exists \nu \in E \, (\nu \notin F))$

but $E$ is, by hypothesis, one of these standard and finite $F$, so it cannot contain such non-standard $\nu$.

A corollary of this theorem is:

Theorem **(8)**:

$$\text{St } x \wedge \exists y \, (y \in x \wedge \text{non-standard } y) \rightarrow \text{Infinite } x$$

"if a standard set contains a non-standard element then this set is infinite"

Proof: it follows immediately from Theorem (7), as a finite and standard set cannot contain a non-standard element. Notice that we are using the set theoretical concept of infinite, that is, Dedekind's.

Subsequently to Theorem (8), we should admit the existence of a (Dedekind) infinite "natural number" if we have reasons to believe that there exists *a standard natural number with non-standard predecessors*. (We cannot expect to have an internal proof of this if the natural numbers are defined as, precisely, the finite ordinals). Let us try to define a "number" that we should interpret as having non-standard predecessors.

The number "$\mu_0$" (mu sub-0), $\mu_0 = 10^{10^{\cdot^{\cdot^{10}}}}$ (10 times) would be an even better candidate than Lavine's to be considered as practically infinite[xix], it is standard (as it is classically defined), does it have non-standard predecessors?

Neither $\mu_0$ nor any other number is big enough to contain elements that must be considered inaccessible, not in practice, but in theory. Nonetheless the theory forces us to believe that there cannot be an infinity of standard natural numbers, so non-standard numbers should appear, even theoretically, after some finite number of "+1" steps. Would they appear before some natural number we can define (and that therefore is standard)? I suggest taking "theoretically inaccessible" as equivalent to being inaccessible not only here and now, but also in the future, and by any mathematician in our universe, human or otherwise: we need a more indisputable candidate. So let us use a few lines to define a larger one. Consider the following series, starting from our $\mu_0$:

$$\mu_0, \ \mu_1, \mu_2, \ldots$$

which elements are defined recursively:

Definition: $\mu_{n+1} = 10^{\cdot^{\cdot^{10}}}$ ($\mu_n$ times) e.g.,

$$\mu_1 = 10^{\cdot^{\cdot^{\cdot 10}}} {}_{(\mu_0 \text{ times})} = 10^{\cdot^{\cdot^{10}}} {}_{(10^{\cdot^{\cdot^{10}}} \text{ (10 times)}} \text{ times)}$$

Now, using this succession we can define an absurdly large number (if $\mu_1$ is not): let us consider "$\mu_{\mu_1}$", the $\mu_1$th element of the succession: $\mu_{\mu_1}$ must be an already big number; but far smaller than $\mu_{\mu_{\mu_1}}$, the $\mu_{\mu_1}$th element of the succession. Let me stop here and define my candidate, "$\mu_s$" (mu sub-s):

Definition: $$\mu_s = \mu_{\mu_{\cdot_{\cdot_{\cdot_{\mu_{\mu_1}}}}}} \text{ } (\mu_1 \text{ levels})$$

As $\mu_s$ is classically defined[xx], it is standard, but it seems quite obvious that almost all its predecessors must be impossible to define (or to count up to) in our real world: can we say, notwithstanding, that all the predecessors of $\mu_s$ are standard? The new axioms of IST cannot help us in deciding which is the case: as we cannot say that all the predecessors of $\mu_s$ are (classically or not) defined, we cannot apply either (T) or (S) to decide that they are standard, although neither can we establish that they are non-standard, even if many of them seem to be non definable.

In [Diener 1989] it is proposed to consider *standard* as precisely equivalent to "definable" (here understood as "definable classically"), but *standard* is really a formal theoretical concept (defined implicitly by the three new axioms) that cannot be reduced to other pre-existing concepts[xxi]; we cannot interpret *standard* as formally representing, in particular, the property of being *definable*.

Nevertheless, if we limit ourselves to the natural numbers, it may be (meta) argued that there cannot be *standard indefinable* numbers, because a number can always have a definition of its length (e.g. as it can be reached by the succession 0, 0', 0'', 0''', ... any number, *n*, can be defined "extensively" by "$n = 0''^{\cdots}'$", where " ' " is repeated *n* times) and if a number, say σ, is standard, then any definition of length σ or shorter would be of standard length, and we must admit that a number with a definition of standard length is standard. For natural numbers, in spite of the fact that "definable" is not a formal concept [xxii], internal to IST, we can take *standard* as representing it; and, as we may well interpret that "only what is *definable* is *available* to us" (as a concrete and separate element we can talk about, and not only as a member of an available set), we may take *standard* as representing also *available.* But if a number is non-standard if and only if it cannot be defined, then many numbers before $\mu_s$ must be seen as non-standard, because impossible to define[xxiii].

If we admit that $\mu_s$, a standard set, has a non-standard element (predecessor), would then $\mu_s$ be an infinite natural number? It would be infinite, according to Theorem (8), but it would not be a natural number ... formally speaking, that is, according to any of the usual definitions of *natural number* in set theory, because only the finite ordinals are considered natural numbers. But $\mu_s$, being the result of a classically defined operation, should be considered a Peano number, an element of *N.* It is in this sense that I would say that $\mu_s$ is an "infinite natural number" in spite of being, with our supposition of having a non-standard predecessor, (Dedekind) infinite and, therefore, not a formal von Neumann natural number. We can, once it is accepted that it has non-standard predecessors, define a self-pairing including all the standard elements of $\mu_s$ (some of them must be infinite): $n \rightarrow 2n$ for finite $n \in \mu_s$ and $\nu \rightarrow \nu$ for infinite $\nu \in \mu_s$ (notice that this is now an internal self-pairing because not using *standard* ).

We cannot take in IST, $N = \omega$, because $N$ contains infinite elements; the **Z**–chains will contain also infinite elements and be themselves infinite; we will have any cardinality of **Z**–chains and therefore we must take to be $N = \Omega$. What happens with the property of the finite ordinals being closed under the addition operation? $\mu_s$ should be considered as the result of repeated "+1" additions, but not of a finite number of them: the property continues being valid for a finite number of repetitions of "+1". $\mu_s$ will not be accessible "from below" (counting, that is, adding "+1" repeatedly), but it will be accessible directly through definition; re-starting from it we can then define a series of predecessors:

$$\mu_s -1, \mu_s -2, \ldots \quad \text{(including, } \mu_s/2\text{, etc.)}$$

those "natural numbers" will also be standard (classically defined) and infinite (will have non-standard predecessors). Can we combine them with the series of finite standard natural numbers in a single series? That is:

$$0, 1, 2, \ldots, \mu_s -2, \mu_s -1, \mu_s$$

In the meta-theory we see $\mu_s$ as a natural number and the series as reaching it through continuous "+1" increments, but in the formal theory $\mu_s$, as well as any predecessor defined from it, is seen as coming after a "gap" of inaccessible numbers that in IST must be considered *external*: the numbers in the gap cannot be members of the theory because they do not comply with the due properties (e.g., some number will be the non-standard predecessor of the minimum infinite standard –namely $\omega$; we may redefine $\omega$ from $\mu_s$ as, $\omega = \{x \in \mu_s$: finite $x\}$).

IST respects Bivalence, the numbers becoming gradually infinite only inside the penumbra zone provided by the external elements; such zone is internally a gap and the series is seen as combining two independent and disjoint series:

$$0, 1, 2, \ldots ; \ldots., \mu_s -2, \mu_s -1, \mu_s$$

one ascending and the other descending, without internal points in common: they are either finite or infinite, the infinite ones not being formally natural numbers (although we "know" they are continuous successors of 0). $\mu_s$ confirms the view that (Dedekind) infinity is a property linked with gaps: the elements after a gap, in a series, can be self–paired, because the (real) impairing cannot be seen inside the gap. What is new here is that a number after a gap, and subsequently infinite, can nonetheless be defined as the product of a finite operation over finite numbers.

$\mu_s$ is not only, as seen, (Dedekind) infinite, it is the inverse of a (Robinson) infinitesimal: as, for any natural number $n$, standard but without non-standard predecessors, $\mu_s/n$ will be definable but still big enough to have non definable predecessors, its inverse, $n/\mu_s$, will be smaller than any $1/n$, implying that $1/\mu_s$ must be taken as being (Robinson) infinitesimal.

However, in IST the infinitesimals are the inverses of finite non-standard unlimited, not the inverses of standard infinite "numbers", so where does $\mu_s$ fit? Its inverse, $1/\mu_s$ , will be smaller than any infinitesimal; contrary to NSA where the infinitesimals approach 0 endlessly as the cardinality of *$R$ approaches that of the proper classes, in IST we can define a number smaller than all its infinitesimals: the inverses of finite unlimited seem not to be small enough to be "real" infinitesimals (smaller than any definable, and therefore standard, number).

To solve this limitation we should redefine "infinitesimal" in IST in order to include numbers like $1/\mu_s$ , or, better, redefine the "unlimited" to include $\mu_s$ : we will then have finite as well as infinite unlimited, and standard as well as non-standard infinite numbers and infinitesimals. That departs from NSA and IST, but is what we seem to be forced to admit by the theorems and arguments above. A *limited* will then be a real smaller than some standard *and finite* real number (the limited non infinitesimal being the *observable*). We will then have the following categories of positive natural numbers and their inverses, {0, 1, 2, 1/2, 3, 1/3,..., $\nu$ –1, 1/$\nu$ –1, $\nu$, 1/$\nu$, $\nu$ +1, 1/$\nu$ +1,...}, ordered by increasing size:

zero (st.) – infinitesimal (st. and non-st.) – observable (st.) –
– unlimited finite (non-st.) – unlimited infinite (st. and non-st.)

Such kind of redefinition (like the one in the next section) will make IST–AI a better candidate for being the unified theory.

## 16. Basis for a unified theory: *IST-AI* with Conway's *numbers*

According to all that has been said, an ideal set theory, intended to be the foundation for all Mathematics, should include the following requirements:

(a) An infinity in accordance with our experience of the indefinitely large and in agreement with the consequences of the finiteness theorems we have seen (Mycielski's and Theorem (5) ).

(b) The demonstrability power of Non-Standard Analysis added to that of Set Theory.

(c) Infinitesimals –with their full structure (**Z**-chains, etc)– as the inverses of the transfinite numbers.

(d) A number system able to provide a measure of quantity for the infinite (and the infinitesimal) sets equivalent to the one we have for the finite.

IST–AI solves (a), (T) being, as seen when combined with (I), equivalent to a formalization of Mycielski's Theorem. IST–AI also solves (b), being an addition of Non-standard Analysis to ZFC.

But IST–AI, in its usual interpretation, does not solve (c) because it continues to contain two disjoint number systems, its infinitesimals being the inverses of finite unlimited numbers and not the inverses of transfinite ones: the very interesting structure of the transfinite, discovered by set theorists, cannot be of any help in establishing the structure of the infinitesimals of IST. Is it possible to have a kind of infinitesimals whose inverses coincide with the usual transfinite numbers? That is what we need in order to have a theory combining both classes of numbers, not just adding one class beside the other, the way IST does.

The problem in order to have the infinitesimals as inverses of the transfinite ordinals is that the transfinite contain only a "single infinity":

$$\lambda, \lambda +1, \lambda +2, \ldots \quad (\lambda \text{ a limit ordinal})$$

while the infinitesimals and their inverses require a "double infinity" of the type:

$$\ldots \infty -2 , \infty -1 , \infty , \infty +1 , \infty +2 , \ldots$$

in which we have infinitely descending chains as well as infinitely ascending ones.

In spite of the fact that it seems that a double infinity is not possible inside a well-founded theory like ZFC, because of the infinitely descending chains, the feasibility of having a number system inside ZFC not only complying with this requirement of double infinity, but also effectively containing infinitesimal numbers as inverses of infinite ones (and not of unlimited) has been established with the construction of some brilliant samples, like the system of *surreal numbers* presented by **John H. Conway** [2001 (first ed. 1976)].

Conway defines a new number system that includes not only the traditional natural numbers, integers, rational and real numbers (obtained usually through successive expansions) but also, in one single system, the transfinite and infinitesimal numbers. The presentation of the system goes as follows [Conway 2001, pp. 4, 5] (I write "*number*" –always in italics– for the new numbers. The definitions are inductive):

**"Construction**

If $L$, $R$ are any two sets of *numbers*, and no member of $L$ is ≥ any member of $R$, then there is a *number* $\{L \mid R\}$. All *numbers* are constructed in this way.

**Convention**

If $x = \{L \mid R\}$ we write $x^L$ for the typical member of $L$, and $x^R$ for the typical member of $R$. For $x$ itself we then write $\{x^L \mid x^R\}$.

$x = \{a, b, c, \ldots \mid d, e, f, \ldots\}$ means that $x = \{L \mid R\}$, where $a, b, c, \ldots$ are the typical members of $L$, and $d, e, f, \ldots$ the typical members of $R$.

**Definitions**[xxiv]

'$x \geq y$' if (no $x^R \leq y$ and $x \leq$ no $y^L$); '$x \leq y$' if $y \geq x$
'$x = y$' if ($x \geq y$ and $y \geq x$)
'$x > y$' if ($x \geq y$ and $y \geq x$ does not hold); '$x < y$' if $y > x$

'$x + y$' $= \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$
'$-x$' $= \{-x^R \mid -x^L\}$
'$xy$' $= \{x^Ly + xy^L - x^Ly^L, x^Ry + xy^R - x^Ry^R \mid x^Ly + xy^R - x^Ly^R, x^Ry + xy^L - x^Ry^L\}$

It is remarkable that these few lines already define a real-closed Field with a very rich structure.

(...) A most important comment is that *the notion of equality is a defined relation*. Thus, apparently different definitions will produce the same number, and we must distinguish between the *form* $\{L|R\}$ of a *number* and the *number* itself."

Considering the above, a set *x* is a *number* if, and only if:

(1) it is an *ordered pair* (the set { {*L*} , {*L*, *R*} } ),

(2) obtained in some of the successive "ranks" of the constructible universe of sets in such a way that

(3) *L* and *R* are sets of *numbers* (being, or not, themselves a *number*) of a lower rank, and such that

(4) $L < x < R$ , understood as *x* not being smaller (larger) than any element $x^L$ of *L* ($x^R$ of *R*).

The "first" *number* to be constructed is the *zero* or " *0* " (in italics to distinguish the new *numbers*):

$$0 = \{ | \} = \{ \{\{ \}\} , \{\{ \} , \{ \}\} \} = \{ \{\{ \}\} \}$$

*0* appears in **P** (**P** (**P** { }) ), that is, in "rank" R(3)[xxv]. Conway calls these ranks "day 0". *0* exists (is a set of ZFC) ex Axiom of Pairing (a theorem in some presentations, saying that "if *a* and *b* are sets, so is {*a, b*}"), the existence of the elements of the pair, the void set or { }, being already derived from the axioms. Pairing assures then the existence of further pairs of previous *numbers*.

It can be shown that the operations defined have all the usual properties, including unrestricted inverses, the resulting number system constituting a Field that contains not only *numbers* "isomorphic" to the real numbers [Knut 79, p. 99] –which include natural numbers, rational numbers, etc.– but also elements obviously equivalent to the transfinite ordinals and to the infinitesimals and their infinite inverses. Let us see a collection of samples (I have added their rank –if two *numbers*, *n*, *m* , both appear in R(n), then $\{n|m\}$ will appear[xxvi] in R(n +3) – "day n"):

| | | | |
|---|---|---|---|
| $-1 = \{ \| 0 \}$ | $1 = \{ 0 \| \}$ | R(6) | day 2 |
| $-1/2 = \{ 1 \| 0 \}$ | $2 = \{ 1 \| \}$ | R(9) | day 3 |
| $-1/4 = \{ 1/2 \| 0 \}$ | $3 = \{ 2 ? \}$ | R(12) | day 4 |
| ... | | ... | |
| $\omega = \{ 1,2,3, ... \| \}$ | | R(ω +3) | day ω [xxvii] |
| $-\omega = \{ ?1,2,3, ... \}$ | | R(ω +3) | day ω |
| $\pi = \{ 3, 3.1, 3.14, 3.141, ... \| 4, 3.2, 3.15, 3.142, ... \}$ | | R(ω +3) | day ω |
| $\iota = \{ 0 \| 1, 1/2, 1/4, 1/8, ... \}$ | | R(ω +3) | day ω |
| … | | … | |

The (equivalents of the) *ordinals* will have the *form* $\alpha = \{L|\}$ [Conway 2001, p. 27]; a *natural number* will be an *ordinal* appearing before day ω. None of the *numbers* is (Dedekind) infinite because, as any ordered pair, they only have two elements, but $\omega$ is larger (in the usual arithmetical sense) than *1,* than *2*, than *3*, etc. and is therefore (Robinson) infinite; a suitable redefinition for "*(Dedekind) infinite*", extended also to the *infinite negative numbers* such as $-\omega$ (smaller than, *–1, –2, –3, …*), could be "*infinite x* if for any *natural number, n* : $x > n$ or $x < -n$", if not it is "*finite*" (based on a definition on p.160 of [Gomez 1989], where Victor Gomez Pin gives a Philosophical foundation for the infinitesimals). Subsequently, for a *number*, being (Robinson) infinite is equivalent to being *(Dedekind) infinite*.

In day ω we will have all the (equivalent) *real numbers* defined as ordered pairs, similar to the definition of π above –that is, in general, as an ordered pair of infinite sets whose elements approach the real by defect and by excess. We have in day ω, as seen, $\iota$ (iota), bigger than *0* and smaller than 1/*n* for any *natural number, n,* a property which we can take (extended to *negative numbers*) as a definition of being "(Robinson) infinitesimal". In day ω we are going to have also some other *hyperreal numbers*: that of the form $d \pm \iota$ , where *d* is any of the *real* that do appear before R(ω) (those that can be written with a finite number of decimal figures in base 2).

In R(ω +6), "day ω +1", will appear $\omega +1 = \{\omega|\}$, also $\iota /2 = \{\ 0 \ | \ \iota\ \}$ (a (Robinson) infinitesimal smaller than $\iota$) and all the *hyperreal numbers* of the form $r \pm \iota$ for any *real, r* ; but we will need to go up to R(ω + ω +3), "day ω.2", in order to have the full series of *hyperreal numbers* of first order, $r\ \iota/n$ (it is worth recalling that, as said in (Kunen 1990, 2.9 Lemma): " *Z*, *Q*, *R* and *C* are all in R(ω + ω)"). But there are more *hyperreal numbers*: at day ω.2 will also appear $\iota^2 = \iota.1/\omega = \{\ 0 \,|\, \iota/2,\ \iota/4,\ \iota/8,\ ...\}$ smaller than any of the *positive numbers* available in previous ranks; $\iota^2$ is what Euler called an "infinitesimal of second order" (*Opera Omnia,* X, 73-75), an infinitesimal you can neglect in your calculus in front of an infinitesimal of first order, something fundamental in Infinitesimal Analysis.

The fact that $\omega,\ \omega +1,\ \omega +2,\ ...\ ,\ \omega +\omega,$ *etc.* form a series isomorphic to that of the transfinite ordinals (for each ordinal α we have the corresponding *ordinal* $\alpha$ at R(α +3) ) implies that the cardinality of **No** (the class of all *numbers*) is the cardinality of Ω, that of the proper classes; and as all the inverses of *ordinals* are *infinitesimals*, the class of the infinitesimals has to be also proper –a good reason to believe that such *numbers* provide, as in NSA, an ultimate division of the continuum.

Returning to the ranks, at day ω +n ("n" whatsoever finite ordinal) we will also have some less usual *numbers*, like $\omega -1 = \{1, 2, 3, ... \,|\, \omega\}$, $\omega -2 = \{1, 2, 3, ... \,|\, \omega -1\}$, $\omega -3, ...$ , all of them *numbers* bigger than any *natural number* (and, therefore, (Robinson) infinite) but smaller than $\omega$. The descending chain, $\omega,\ \omega -1,\ \omega -2,\ ...$ , does not contradict Foundation because $\omega -1 \notin \omega$, $\omega -1$ is not isomorphic to any *ordinal*, $\alpha$ , such that $\omega = \alpha' = \alpha \cup \alpha$ : $\omega -1$ is not the predecessor of $\omega$, in the ordinal sense, although it is its "*predecessor*" in the sense that $\omega = (\omega -1) + 1$ (Conway's addition is not the repetition of the successor operation, although we can call $n +1$ the "*successor*" of *n*). As seen with $\mu_s$, and for the same reasons, we cannot consider formally that there is a continuous series, $0,\ 1,\ 2,\ ...\ ,\omega -2,\ \omega -1,\ \omega$, but two independent ones, even though we can admit that $\omega<\mu_s$ ($\mu_s$ isomorphic to $\mu_s$ ) and then "see" the series as continuous.

We will also have more bizarre elements, like $\omega+1/2$ , $\omega/2$ , $\sqrt{\omega}$ or $\sqrt{\iota}$ , that confirm that we have available all the usual operations on the new *numbers*, no matter if they are *infinite* or *infinitesimal* or whatsoever combination of both. More than enough to fulfill requirement (c), because those examples show that we can define, inside Conway's *numbers*, the usual tools of NSA, like the **Z**-chain, or *galaxy*, of any infinite number (the galaxy of an infinite $\nu$ will now be $\nu \pm n$, $n$ each *finite natural number*), or the *standard part* (will now be the standard *real r* such that $r = x \pm í$, *í infinitesimal*), etc.

Conway tells us [2001, p. 44]: "We can of course use the Field of all numbers, or rather various subfields of it, as a vehicle for the techniques of the non-standard analysis developed by Abraham Robinson. Thus for instance for any reasonable function *f*, we can define the derivative of *f* at the real number *x* to be the closest real number to the quotient

$$\frac{f\,[x + (1/\omega) - f\,(x)]}{1/\omega}$$

"The reason is that *any* totally ordered field is a model for the elementary statements about the real numbers."

Anyway Conway does not recommend using his system as a source of infinitesimals [continuing the same quotation]: "But for precisely this reason, there is little point in using subfields of **No** when so many more visible fields will do. So we can say that in fact the Field **No** is really irrelevant to non-standard analysis." A not so minor point is that those "more visible fields" do not provide the infinitesimals as inverses of the (equivalents of) ZFC's transfinite ordinals, and, subsequently, do not avoid having two disjoint number systems. Another big point is that an omni-comprehensive system seems better than a less powerful one. Additionally, Conway's system allows treating the infinities "Euler style", as can be seen with the "infinite sums" in [Conway 2001, pp. 39-40]. (Nevertheless, a subfield with easier to use operations would be welcome).

The proposal is to redefine IST–AI in order to use (some subfield of, or any system equivalent to) Conway's *numbers* as a source of transfinite (the "*unlimited*" being then redefined as any *number* larger than the *finite* and standard), as well as of infinitesimal numbers (an "*infinitesimal"* would be the inverse of an *unlimited*). But then, would the new *infinitesimals* allow all the proofs and elements necessary for Calculus without needing to introduce the *standard / non-standard* distinction? In other words, could we content ourselves working with Conway's *numbers* inside ZFC or should we work in IST (more precisely, in IST–AI)? If the only reason for introducing the "standard / non-standard" distinction is to have the infinitesimals, as their function is perfectly fulfilled by the *infinitesimal numbers*, we do not need the axioms defining implicitly the concept "standard" (nor do we need to use ultraproducts or other model theoretical methods).

But there are some good reasons for preferring IST–AI to ZFC as a foundation for Mathematics:

- AI can be deduced from (I) and (T): non-standard analysis provides a foundation not only for infinitesimals, but also for the infinite

- as argued, (I) and (T) are more obvious axioms than AI ( (S) being simply an extension of Comprehension);

- we can find for the (I), (S) and (T) axioms other uses independent from the infinitesimals, like obtaining conclusions about the composition of infinite sets (we have already seen how they are composed of a finite number of standard elements plus an infinite number of non-standard elements), clarifying the content of the vague zone we represent by the "...". No doubt the standard / non-standard distinction can provide much more (non classical) conclusions about the composition of infinite sets, affording a new insight into infinity;

- while we are obliged to interpret that there should be a gap before $\omega$ (as a predecessor ordinal of $\omega$ would be contradictory), and that formally there is also an equivalent gap before $\omega$, the (double) series, *0,1,2, ... ; ... , $\omega -2$, $\omega -1$, $\omega$,* may perfectly well be interpreted as really continuous (without gaps), changing gradually from finite to infinite. The elements external to IST allow such a gradual conversion, as is the case with the predecessors of $\mu_s$ ;

- IST–AI is a theory equivalent, in the sense explained, to *Fin* (ZFC), allowing the "too large to count" interpretation of infinity in agreement with the finiteness theorems;

- many proofs of theorems about infinitesimals have shown to be much easier inside IST; although, in theory, they can also be proved inside ZFC, this may not be feasible in practice. Easier proofs should also be expected for theorems about infinite sets;

- last but not least, the introduction of the "available / non-available" distinction (as we may interpret standard / non-standard) brings Mathematics closer to Physics and to the real world: infinities, understood as "too large to apply the same laws", may well be admitted in Physics as a means to distinguish a level of reality (the macro-world) for which the laws of the finite level (the micro-world) are not well suited (although we know that they would be valid if we were able to analyze "too large to count" quantities of elements).

## 17. *Quantal* (size) numbers

Even if you dislike using IST–AI instead of ZFC, the latter can no longer be interpreted under the Cantorian view after having seen its failure in Finite and Non-Standard Analysis. But if cardinals can no longer be used to measure the quantity of elements in a set under the Euclidean interpretation, how can we measure it? The main function of a number system is to be used to measure something, to represent the quantity of something. Is the Conway *number* system the one we promised would be able to measure the quantity of elements in a set, thus fulfilling requirement (d)?

The "size", or number of elements, of a finite set is already measured by the finite natural numbers. Any class of sets complying with Peano axioms can be called "the natural numbers"; the preferred such class is that of the finite von Neumann ordinals, but the new *natural numbers* can also be seen as complying with such axioms and, therefore, to measure the size of a finite set will be equivalent to using Conway's or von Neumann's numbers.

Under the Cantorian interpretation, the size of an infinite set is measured by the smallest ordinal bijectable with this set, that is, by its cardinal: Conway's *infinite numbers* cannot, under this view, measure size (as cardinality) of sets.

But under the Euclidean interpretation (considering that, for example, the set of even numbers has to have fewer elements than the set of natural numbers, of which it is a proper part) the infinite von Neumann cardinals cannot provide a measure of the size of a set like that of the even numbers (we should, therefore, to avoid confusion, no longer call the size "cardinal": let us call the size "***quantal***") because there is not any infinite cardinal smaller than ω (the *quantal* of the set of all the natural numbers) and the *quantal* of the set of the even numbers must be infinite but smaller than the set of the naturals. Inside the new Conway *numbers*, on the contrary, we have *infinite numbers* smaller than $\omega$ that can provide a measure for such a size, although we will need to "compute" in some way the size, as pairing no longer can help us in deciding the size of infinite sets (because of the possibility to pair sets of different sizes), and we do not have, for the moment, any alternative method to decide the *quantal* of an infinite set.

Anyway, it seems intuitively that if the set $\omega = \{0, 1, 2, ...\}$ has quantal $\omega$ (we can put "*quantal* $\{0, 1, ...\} = \omega$"), then *quantal* $\{1, 2, 3, ...\} = \omega - 1$, the latter set having one element "less" than the former, the 0; and, as "half" the natural numbers are even and "half" odd (because of each two successive naturals, one is even and one is odd), then it seems that we should admit[xxviii]:

$$\textit{quantal}\ \{0, 2, 4, ...\} = \textit{quantal}\ \{1, 3, 5, ...\} = \omega/2 = \{1, 2, 3, ... \mid \omega - 1, \omega - 2, \omega - 3, ...\}$$

*Quantal* $\mathbf{P}\omega$ has no interest because $\omega$ has only two elements, so let us consider the fundamental *quantal* $\mathbf{P}\omega$: its elements can be seen as the permutations of $\omega$ elements taking two possible values –belonging or not belonging to a given subset of ω– the number of elements of such a permutation is, in the case of a finite $n$ number of elements, $2^n$, the new *numbers* being isomorphic to those of the elemental Arithmetic in the finite case. Applying Leibniz's Principle, or if you prefer, taking into account that Conway's operations are exactly the same for *infinite* as for *finite numbers*, the same computation should be valid for a *number* $\omega$ of elements, that is:

$$\textit{quantal}\ \mathbf{P}\omega = 2^{\varpi} \qquad \text{(a uniquely, although not easily, defined } \textit{number}\text{)}$$

What, then, is *quantal* $\aleph_1$ ? $\aleph_1$ is an ordinal, the smallest not enumerable (not bijectable with $\aleph_0 = \omega$), and therefore *quantal* $\aleph_1$ must be much, much larger than $2^{\omega}$ (a *number* smaller than $\omega^{\omega}$) because even

$$\varepsilon_0 = \omega^{\omega^{\cdot^{\cdot^{\cdot}}}} \text{ (ω times)}$$

and other much, much larger ordinals[xxix] are enumerable. That should not be a surprise, because, with the reinterpretation of cardinality already presented, cardinality is unrelated to size. It is the combinatorial nature of its elements, and not its *quantal*, that makes $\mathbf{P}\omega$ non-denumerable.

IST can help in explaining why $\mathbf{P}\omega$ can be demonstrated to be non-denumerable in spite of its relatively small *quantal*: in any practice only the subsets of ω contained in a finite subset of $\mathbf{P}\omega$ (its standard subsets) are going to be defined, and therefore it seems theoretically possible to have a pairing of these standard subsets with the standard natural numbers (all the subsets could then be said to be paired); why, in this case, could the fact that the pairing is not really onto (as $2^{\omega} > \omega$) not be hidden inside the vague zone? Because the Liar-like diagonal element (which like whatsoever subset of ω must be in $\mathbf{P}\omega$) would be classically defined (if we were to have a defined bijection, the conditional hypothesis on Cantor's Theorem), and therefore standard, and so its correlate should also be standard and, subsequently, we could not hide this standard pair inside the non-standard zone.

Let me conclude with a corollary with meta-physical consequences: it can be shown that [Conway 2001, p.7]:

$$0\nu = 0 \qquad \text{for any } \textit{number } \nu$$

confirming the classical assertion that "adding nothingness, no matter how many times, you cannot have but nothing". And therefore, no matter how large $\nu$

$$1/\nu > 0$$

you cannot arrive at *0* by dividing: for *infinite* $\nu$, the *infinitesimal* $1/\nu$ becomes endlessly small as *n* becomes larger and larger, nevertheless, the limit for $\nu$, $\Omega$, is not a *number*, but a proper class, a Cantor's absolute infinity that you can only potentially approach; the absolutely infinite division (the only one that could reach *0*) is contradictory, as absurd as having the set of all *ordinals*. Being impossible even mathematically, any process of successive divisions in the real world must come to an end; there must be an absolutely minimum size. Achilles could not overtake Zeno's Turtle if there was not a minimum, non-zero, size in his Universe.

## Notes[xxx]

i D. A. Steele, in the Introduction to his English translation of B. Bolzano's *Paradoxes of the Infinite*, says that this paradox was already known to Proclus and Plutarco, although until Bolzano it was not presented as a property of infinite collections.

ii In [Soler, Soler and Soler 1999, pp. 47-50] we simply named it "the third solution".

iii See, for example, [Soler, Soler and Soler 1999, pp. 314-320].

iv As we are reminded in [Jané 1993, p.71]: "Since the arithmetical operations of addition, multiplication, exponentiation and so forth are definable by second-order formulas from the successor operation, among the logical consequences of these <Peano> axioms there must be solutions to all open problems of elementary number theory." W. Ackerman, noticing that there is no reason to stop at exponentiation, defined an endless series of operations, each one abridging the repetition of the preceding repetition, [Ackerman 1956, pp. 336-345].

v Anyway, $\aleph_{\alpha} (+) \aleph_{\beta} = \aleph_{\alpha+\beta}$ , is a no-nonsense result that would allow us to move along the $\aleph$-function.

vi If $n$+1 was infinite, there will be a pairing of $n$+1 with a proper set, which pairing could be modified by suppressing from it the one or two elements (ordered pairs) that contain $n$ as an element of the ordered pair, obtaining a pairing of $n$ with a proper subset of $n$, contrary to the hypothesis.

vii The most classic version is the successful *fuzzy logic* introduced in [Zadeh 1965].

viii The author of the paradox; without limits in the original version.

ix I present a version (as usual from [Lavine 1994], here pp. 270-1), adapted for set theories, with "$a \in \Omega_p$" instead of "$\Omega_p(a)$". If the theory uses equality, a regular relativization of the axioms of equality must also be considered included.

x Axiom (**1**): for any constant symbol, $c$, in the vocabulary of $T$: $c \in \Omega_p$

Axiom (**2**): for p < q and for any n-function symbol $f$ : $(\forall x_1 , \ldots , x_n \in \Omega_p) f(x_1, \ldots , x_n) \in \Omega_q$

Axiom (**3**): for p < q : $(\forall x \in \Omega_p)\ x \in \Omega_q$ .

They regulate (informally speaking): Axiom (1) that any object denoted by the constant symbols must be in $\Omega_p$; Axiom (2) that, if $a_1, a_2, \ldots , a_n$ are in $\Omega_p$, then $f(a_1, \ldots , a_n)$ must be in any set indefinitely large with respect to $\Omega_p$ ; Axiom (3) that $\Omega_q$ must include $\Omega_p$ if p < q (it could be a Corollary of Axiom (2), taking $f(x) = x$).

xi In *Fundamenta Mathematicae* **23**, 150-151,

xii Where [Theorem 5, p. 10] the existence of a kind of non-absolute sentences (not valid in the ground model but valid in some extension) is proved, or in [Bagaría 1977, p. 366-72], where it is proved that: "The absoluteness under ccc forcing extensions of all $\Sigma_1$ sentences with $A \subset \kappa$ , $\kappa < 2^{\aleph(0)}$, as a parameter (...) is equivalent to Martin's axiom".

xiii I will follow the version of IST presented in [Robert 1985, Chapters 2 and 3] but maintaining the name "internal", used by Nelson and Robinson, in addition to "classical" used by Robert.

xiv (T) is not valid for non classical formulas. A formula using "standard" may well be valid for the standard and not valid for the non-standard: if $\nu$ is a non-standard natural number, then any finite standard natural number will be smaller than n, but that cannot imply that all the non-standard are smaller than $\nu$.

Transfer reassures us that all ZFC is valid in IST, although it seems to me that we must consider implicit Robinson's requirement that [1974, p. 49]: "**3.1** (ii) Every mathematical statement which is meaningful and true for the system of the Natural Numbers is meaningful and true also for $^*N$ : provided that we interpret any reference to entities of any given type, e.g., sets, or relations, or functions, in $*N$ not in terms of the totality of entities of that type, but in terms of a certain subset, called the set of *internal* entities of that type. (...) (iii) The system of internal entities in $*N$ has the following property. If $S$ is an internal set of relations, then all elements of $s$ are internal." We have seen that a non-standard could be external, in which case a sentence of ZFC could be valid, not "for all non-standard", but only "for all internal non-standard".

Alain Robert [1985, solution to exercise 1.9.1] says: "When proved for all $n \in N$, can anyone doubt its validity for nonstandard $n$?" I doubt, according to the quoted paragraph from Robinson, that this is valid for the external non-standard $n$: ZFC's statements do not tell us anything about an external $n$ ($n \notin *N$). In particular, the induction principle: $P(0)$ & ( $P(n) \rightarrow P(n+1)$ ) $\Rightarrow \forall n\ P(n)$, could not imply $P(x)$ for external non-standard $n$. Interpreted in the sense that (T) implies that the classical statements are valid for all the non-standard (external ones included), (T) will not be equivalent, as intended, to the property required in (Robinson 1974, 3.1 (ii) ), where such statements are said to be valid only for the *internal* sets, which, according to Robinson, do not include all the non-standard.

xv My own proof, as far as I know.

xvi It does not appear in [Robert 1985], it is Theorem 1.2 in [Nelson 1977].

xvii Anyway, before obtaining any strong finiteness conclusion from theorems (4) and (5), it must be noticed that their proofs are based in the bivalent conclusion that "the sets are either finite or in-finite (not finite)": under a gradualist view, with those standard and finite subsets converting gradually to infinite as much as to non-standard, the existence of a finite element (or of a finite set) bigger than all the elements (or containing all the elements) of each standard and finite set could not be admitted. Notwithstanding, the proofs would remain valid if we were to consider that all the standard are (fully) finite because the gradual conversion to infinite takes place only inside the non-standard zone, a highly plausible hypothesis; for this reason, from now on I follow, unless otherwise stated, the usual bivalent view, but it is worth recalling that it is a simplification.

xviii Robinson's NSA is said not to be well-founded (see, for example, [Kunen 1990, p. 145] ), based on the existence of *(Robinson) infinite* descending chains of non-standard elements. However, this does not introduce a contradiction with the Axiom of Foundation (or Regularity) because these elements are non-standard. Nelson affirms that IST is well-founded: "Let $n$ be an unlimited natural number in a model of IST. Then $n$, $n$-1, $n$-2, $n$-3, ... is an external infinite sequence that each term is $\in$ its predecessor. The axiom of regularity precludes the existence of such a sequence (...) Within IST, of course, the sequence is finite and terminates after $n$ steps" [Nelson 1977, p. 1197]. In spite of the fact that there are *unlimited* descending chains, these are not *infinite*, we will have just an element of the chain for each standard $n$, departing from any non-standard $\nu$.

xix There is a usual confusion with such towers of exponentiation that can be shown in the following example:

$$10^{10^{100}} = (\,(10)^{10})^{100} = 10^{10.100} = 10^{\,(10)^{3}} \neq 10^{\,(10)^{100}}$$

when people write $10^{10^{100}}$ they usually intend to mean $10^{\,(10)^{100}}$, not $(\,(10)^{10})^{100}$, the same convention that I intend to use here.

xx Actually, the definition of $\mu_s$ includes not only the latter "definition" but also all the previous steps to arrive at it. I have been using, implicitly, operations of higher order: for $\mu_0$ the repetition of exponentiation, sometimes called "tetration" [Rucker 1995, pp. 69-99], and for $\mu_1$ , "pentation", the repetition of tetration. In [Soler, Soler and, Soler, II Appendix] a much, much larger natural number is defined using the generalized operations of W. Ackerman.

xxi Transfer, as seen, only implies that "what can be (classically and uniquely) defined is standard" (equivalent to "the non-standard cannot be defined"), but does not imply its contrary: "what cannot be defined is non-standard" (equivalent to "all the standard can be defined"); formally, there could be standard non-definable elements. We cannot interpret *standard* as formally representing, in general, the property of being *definable*.

xxii Definability, if not limited to some specific formal procedure, is an open process difficult to quantify, as shown by Berry's and Richard's paradoxes (in [Kunen 1990, p. 152] such an argument is used to show that: "There is no way to write a formula to express '$x$ is defined by a formula'").

xxiii We cannot establish formally that it is the case that there cannot be definitions of length $\mu_s$, so the argument can be seen as true only in the meta-theory.

xxiv I omit some minor definitions not used here.

xxv Using the notation in [Kunen 1990, p. 95]:
$R(0) = \{ \}$
$R(\alpha +1) = \mathbf{P} (R(\alpha))$
$R(\lambda) = \cup_{\alpha<\lambda} R(\alpha)$ when $\lambda$ is a limit ordinal

xxvi $\{n\}$, $\{m\}$ will appear in R(n +1), subsequently, $\{\{n\}\}$ and $\{\{n\}, \{m\}\}$ will appear in R(n +2) and, finally, $\{n \mid m\} = <\{n\}, \{m\}> = \{ \{\{n\}\} , \{\{n\} ,\{m\}\} \}$ in R(n +3).

xxvii The existence of a "rank ω" is granted by the axioms, including an Axiom of Infinity or others from which AI can be deduced. Notice that the preexistence of ranks implies that the ordinals are more basic, a prerequisite, than the *numbers*; this is not a limitation, the ordinals being the basis of any set in the hierarchy of constructible sets, the core of set theory.

In R(ω) we only have (Dedekind) infinite sets, containing all the possible combinations of elements appearing at some finite level, but not yet ordered pairs containing such infinite sets; in R(ω +1) we will have sets containing those infinite sets, then in R(ω +2) the sets containing the latter, and in R(ω +3), "day ω", we can already have *numbers* with *L* and /or *R* infinite, the largest appearing this day being *ω*.

xxviii This reasoning was presented as early as the 9th Century by Thābit ibn Qurra, one of the "Baghdad Scholars" [Lévy 2000, insert p. 50].

xxix As shown in [Soler, Soler and Soler 1999, pp. 248-51], where transfinite extrapolations of Ackerman's generalized operations are defined and enumerations for the very large ordinals definable with them are described.

## xxx References


[Ackerman 1956] W. Ackerman, *Zur Axiomatik der Mengenlehre*. Mathematische Annalen, vol. 131.

[Bagaría 1997] Joan Bagaría, *A Characterization of Martin's Axiom in Terms of Absoluteness.* The Journal of Symbolic Logic 62 / 2, pp. 366-372.

[Bagaría and Friedman 2001] Joan Bagaría and Sy D. Friedman. *Generic Absoluteness.* Annals of Pure and Applied Logic 86 / 3, pp. 3-13.

[Cantor 1883] Georg Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein matematischer-philosophischer Versuch in der Lehre des Unendlichen*. B.G. Teubner, Leipzig, 1883. Quoted from [Lavine 1994].

[Conway 2001] John H. Conway, *On Numbers and Games*. A K Peters, Ltd. Natick, Massachusetts. (First edition Academic Press, 1976).

[Delahaye 2000] Jean-Paul Delahaye, *L'infini est-il paradoxale en mathématiques?*. Pour la Science, Special Issue, December 2000.

[Diener 1989] Francine Diener and Marc Diener, *Las aplicaciones del Análisis no estandar*. Mundo Científico 89 / 9, pp. 276-285.

[Levy 2000] Tony Levy, *Tahib ibn Qurra et l'infini numérique.* Pour la Science, Special Issue, December 2000.

[Gómez Pin 1989] Victor Gómez Pin, *Filosofía. El saber del esclavo*. Editorial Anagrama, S.A., Barcelona.

[Gómez Pin 1990] Victor Gómez Pin, *El infinito*. Ediciones Temas de Hoy, S.A., Madrid.

[Jansana 1990] Ramón Jansana, *Una introducción a la lógica modal*. Editorial Tecnos, S.A., Madrid.

[Jané 1993] Ignacio Jané, *A Critical Appraisal of Second Order Logic*. History and Philosophy of Logic, **14** (1993), pp. 67-86. Taylor & Francis Ltd.

[Jech 1978] Thomas Jech, *Set Theory*. Academic Press, Inc., San Diego.

[Knuth 1974] D. E. Knuth, *Surreal Numbers*. Addison-Wesley Publishing Company, Reading, Massachusettts.

[Kunen 1990] Kenneth Kunen, *Set Theory, An Introduction to Independence Proofs*. North Holland.

[Lavine 1994] Shaughan Lavine, *Understanding the Infinite*. Harvard University Press. Cambridge, Massachussetts.

[Manzano 1989] Maria Manzano, *Teoría de Modelos.* Alianza Editorial, S.A., Madrid.

[Mayberry 2000] J. P. Mayberry, *The Foundations of Mathematics in the Theory of Sets*. Cambridge University Press. Cambridge. United Kingdom..

[Mosterín 1980] Jesús Mosterín, *Teoría Axiomática de Conjuntos*. I. G. Seix y Barral Hnos. Esplugues de Llobregat. Barcelona.

[Mycielski 1986] Jan Mycielski, *Locally finite theories*. Journal of Symbolic Logic, 51, pp. 59-62.

[Nelson 1977] Edward Nelson, *Internal set theory, a new aproach to NSA*. Bulletin of the American Mathematical Society, 83, pp. 1165-1198.

[Robert 1988] Alain Robert, *Nonstandard Analysis.* John Wiley & Sons Ltd. Original in French, *Analyse non standard*, 1985. Presses polytechniques romandes, Lausanne.

[Robinson 1974] Abraham Robinson, *Non-standard Analysis*. Revised 2nd edition (1st ed. 1964). Reprinted 1996, Princeton University Press, Princeton.

[Rucker 1983] Rudy Rucker, *Infinity and the Mind*. Princeton University Press. Princeton.

[Sainsbury 1995] R.M. Sainsbury, *Paradoxes*. Revised 2nd edition. Cambridge University Press. New York.

[Soler, Soler and Soler 1999] Miguel Soler Sala, Adrian and Miguel M. Soler Selga. *El Infinito y sus Paradojas: de la solución clásica a la no-estandar.* Research Paper, IESE, Universidad de Navarra, Barcelona. (You may request a copy from: miquelsoler@yahoo.es)

[Stroyan 1991] K.D. Stroyan, *Infinitesimal Analysis of Curves and Surfaces*. In Jon Barwise, Ed., *Handbook of mathematical logic*. 7th ed. Elsevier Science Publishers BV, Amsterdam.

[Zadeh 1965] L.A. Zadeh, *Fuzzy Sets*. Information and Control, 8, pp. 318-353.