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Abstract

In this paper, we present a continuous model to optimize multi-echelon inventory management decisions under stochastic demand. Observing that in such continuous system it is never optimal to let orders cross, we decompose the general problem into a set of single-unit sub-problems that can be solved in a sequential fashion. When shipping and inventory holding costs are linear in the stage, we show that it is optimal to move the unit associated with the k-th next customer if and only if the inventory unit is held in an echelon located within a given interval. This optimal policy can be interpreted as an echelon base-stock policy such that the base-stock is initially increasing and then decreasing in the stage. We also characterize the optimal policy when costs are piecewise-constant. Finally, we study the sensitivity of the optimal base-stock levels to the cost structures.

1 Introduction

Inventory in the supply chain can be held in many different locations, from upstream stages (raw material stockpiles, work-in-progress, finished goods warehouses in factories) to downstream stages close to the final consumption (distribution centers, stores). Supply chains that have very predictable demand and controlled production/distribution processes can operate under a *Just-In-Time* (JIT) regime, where no safety stock is needed. In that case, only in-transit inventory is present, unless some additional inventory is needed when a batch process exists. In contrast, when demand is uncertain, safety stock is necessary at different stages to buffer against demand spikes that are costly. Indeed, it is usually costly to not to serve a customer or to do so late, for many reasons: the customer might leave to a competitor, which causes a loss of margin, corresponding to the current purchase and perhaps to the client's future purchases too; a compensation might be offered to the customer for the inconvenience; and the reception of payment might be delayed, and hence should be discounted by the time-value of money. These expenses are often taken into account in inventory decisions by including a back-ordering penalty to ensure an adequate level of service.

Generally, the management of inventory and, specifically, the positioning of safety stock in multi-echelon systems has been an area of industry attention and a prolific field of research since the 1960s. The structure of the optimal inventory policy is well-known when there are no fixed ordering costs: it is an *echelon base-stock policy*, that depends on the problem parameters (demand, costs). From the seminal work of Clark and Scarf [12], it is relatively easy to calculate the optimal base-stock levels numerically. However, due to the complexity of the task, it is often difficult to intuitively understand the result of the optimization. It is also hard to perform sensitivity analysis on the optimal base-stock levels with respect to the model parameters.

Some alternative approaches to overcome these difficulties have been examined over the years. In particular, Axsäter [2] observed that it was possible to decompose the traditional formulation of the inventory problem as a set of sub-problems, each one of them corresponding to a specific customer. In this paper, we apply this approach to provide some new results on the multi-echelon problem, with the specific objective of understanding better the drivers behind the optimal base-stock levels.

For this purpose, we consider a continuous version of the problem. Namely, we assume that *stages are continuous* and that we consider *continuous-review inventory controls.* This means that each inventory unit can be located in an infinite rather than finite set of positions. The inventory manager can then decide to keep the unit where it is or move it towards the customer at every moment. The cost of doing one action or the other depends on the location of the unit. Of course, our continuous model can mimic any discrete system by making the appropriate assumptions on costs, so it provides quite a bit of modeling flexibility. The assumption of having continuous stages is also relatively realistic. Indeed, in practice there are a number of physical locations where inventory can be placed. In today's extended supply chains, such points span almost the entire chain, and the decision to hold an item or ship it downstream can be taken almost everywhere, except perhaps in long-haul maritime transportation.

Using this continuous setting, we provide a new formulation of the multi-echelon inventory problem using the unit-decomposition approach. The optimization is expressed as an optimal control problem, and the solution can be found by solving a differential equation, called the Hamilton-Jacobi-Bellman equation. We characterize the structure of the optimal solution and can provide in some cases a closed-form expression for the cost-to-go functions. This allows us to understand better the drivers of the optimal stock levels and provides the basis for sensitivity analysis. It also has the potential to generate some approximations that can be useful in practice. Finally, our formulation provides a new solution procedure and new algorithms that perform very well when costs are constant or linear (although in general solving the differential equation can be very expensive computationally).

Thus, the paper contributes to the literature in several dimensions. First, it proposes a novel approach to an old problem that is well studied and generates new insights on the structure of the solutions. Second, it characterizes the solutions explicitly when costs vary linearly over the supply chain. Third, it proposes a new solution procedure that exploits the structure of the problem in a new way, and performs quite well in the instances that we consider.

The rest of the paper is structured as follows. §2 reviews the literature relevant to our work. §3 describes the continuous model and the formulation of the optimization problem. §4 characterizes the optimal inventory policy and provides an explicit recursive expression for the cost-to-go function. We then study the sensitivity of the optimal solution to model parameters in §5 and conclude in §6. The proofs are included in the appendix.

2 Literature Review

Our paper is closely related to the inventory management literature on multi-echelon systems. Clark and Scarf [12] first introduced the notion of echelon-stock and showed that an echelon base-stock policy is optimal in an finite periodic review problem if no fixed ordering costs are present. Axsäter and Rosling [6] showed that the optimal echelon-stock policy derived by Clark and Scarf [12] can be replaced by an equivalent installation-stock policy and they furthermore established the conditions when this can be done. Federgruen and Zipkin [16] extended the Clark and Scarf [12] result to an infinite horizon and Rosling [23] showed that the result is also valid for an assembly systems. An alternative derivation can be found in Chen and Zheng [10]. Several extensions to the original model exist, e.g., DeCroix et al. [14] who consider a system with returns, Gallego and Zipkin [17] who consider stochastic lead-times, and Chen and Song [9] who consider a time-varying demand process. Moreover, a number of approximate approaches for near-optimal results under the original or more complicated assumptions have been suggested over the years, see Clark and Scarf [13] or DeBodt and Graves [15] among others. For thorough reviews see Zipkin [28], Axsäter [4] or van Houtum [26].

The present paper focuses on the same traditional problem, but uses a different approach, the so-called *unit decomposition* approach, pioneered by Axsäter [2]. In a few words, this approach uses the fact that in most systems, it is optimal not to cross orders. As a result, one can account for costs unit by unit, instead of period by period, as the traditional literature does. This allows to decompose the original problem into a set of sub-problems that can be solved individually. Earlier papers in this line of work are Axsäter [3], who applies this technique to batch ordering in a two-level system, or Graves [18], who finds the steady-state distribution of inventory in a system with one depot and many sites. The solution approach has been exploited by Muharremoglu and Tsitsiklis [21] to show that echelon base-stock policies are optimal under general assumptions on lead-times. In comparison, the present paper uses this technique in a multi-echelon setting with the purpose of determining the structure of the optimal echelon base-stocks and providing simple expressions for the optimal base-stock levels and the corresponding cost-to-go functions. In addition, the following papers have also used the technique. Muharremoglu and Tsitsiklis [22] find optimal expediting decisions. Janakiraman and Muckstadt [19] characterize the structure of the optimal policy for capacitated two-echelon systems. In a single echelon context, Martínez-de-Albéniz and Lago [20] show that myopic policies are optimal for a general class of non-stationary, correlated demand processes. Yu and Benjafaar [27] show that echelon base-stock policies are optimal with non-stationary, correlated demands and lead-times. Berling and Martínez-de-Albéniz [7] determine the optimal base-stock levels in a single-echelon system with stochastic purchasing price.

Some researchers have studied continuous multi-echelon systems, as we do here. Song and Zipkin [25] analyze a continuous-stage, continuous-demand inventory problem. They use the traditional approach of cost accounting period by period and obtain closed-form solutions when they set the costs appropriately. Most similar to our work is Axsäter and Lundell [5], that formulate the optimization problem with monotonic holding costs and no moving costs, and report some numerical results in a continuous-stage, discrete-demand setting. In contrast, we provide the general structure of the solution, including some explicit expressions for the cost-to-go functions in some cases. Finally, it is worth mentioning that, in a companion paper, The Authors [1] focus on optimal expediting decisions using the same continuous-stage, discrete-demand model as here.

3 A General Model for a Continuous-Stage Serial Supply Chain

3.1 Model Setting

Consider the following standard multi-echelon inventory problem. A manager is in charge of managing the inventory in a serial supply chain. Inventory can be obtained from an upstream location (the highest echelon), and is shipped downstream in order to fulfill the demand (occurring at the lowest echelon). There are costs involved in shipping the inventory, holding the inventory, and in failing to fulfill the demand on time. The manager's objective is to minimize the expected sum of these three costs.

We focus on a supply chain that has *continuous stages* and where one as a result will make *continuous-review decisions.* That is, at any point in time, each inventory unit located in the chain can be either moved downstream towards lower echelons, or otherwise kept in the same location for a little bit longer. The resulting system is one where there are a number of units spread out over the supply chain, some being moved and others not. The system we have in mind is a production/distribution system where a unit is moved closer to the consumer when it is being processed and one can at any time choose to stop processing that unit. However, by choosing the cost parameters intelligently, one can mimic most serial supply chains, e.g., a standard serial system with a number of warehouses to which the units are successively moved.

We index each stage through its position $x \in [0, L]$, which denotes the "distance" to the downstream customer, i.e., the time it takes to ship a unit from x to 0, measured for example in days of transportation. That is, $x = 0$ is the location immediately next to the customer, while $x = L$ is the upstream location, where an infinite amount of raw material is available. We assume that customers arrive at random times, and in particular that demand is Poisson distributed with a constant intensity λ (i.e., inter-arrival times are i.i.d., exponentially distributed). The methodology can be extended to any renewal process, though. All demand that cannot be met immediately from stock on hand (located at $x = 0$) is back-ordered until more goods are available at this location. There is fixed back-order cost $b > 0$ per time-unit per back-ordered unit. The other costs considered are out-of-pocket holding costs $h(x) \geq 0$ for $0 \leq x \leq L$, and "moving" costs $m(x) \geq -h(x)$ for $0 < x \leq L$ (if the item is moved, zero otherwise; the item cannot be moved further at $x = 0$), both per time-unit and per unit. All costs are discounted with a continuous discount rate of $r > 0$. The moving cost $m(x)$ can be interpreted as the value added to the product as it moves forward through the production process. While we do not explicitly include the capital cost of holding inventory in $h(x)$ (only the out-of-pocket holding cost), this capital cost is indirectly incorporated through the payment of $m(x)$: it is approximately equal to $r \int_x^L m(y)dy$ per time unit, where $\int_x^L m(y)dy$ is the amount of cost that has been incurred until that moment. For simplicity, and in coherence with the assumption that the position x is measured in time units, we assume that the speed at which items travel along the supply chain is equal to one distance-unit per time-unit. In fact, it is possible to consider variable moving speeds, as in the companion paper The Authors [1].

Note that both the holding cost, $h(x)$, and moving cost, $m(x)$, can be stage-dependent. Hence, by choosing them carefully (and possibly relaxing the conditions of no negative costs posed for the "true" costs), one can mimic most serial supply chains. Consider for example the traditional serial supply chain with three physical locations where inventory can be held: an upstream one (the supplier at $x = L$), an intermediate one (a distribution center at $x = D$) and a downstream one (a store at $x = 0$). In our model, this system can be represented by choosing

$$h(x) = \begin{cases} 0 & \text{if } x = L \\ h_D + M & \text{if } D < x < L \\ h_D & \text{if } x = D \\ h_S + M & \text{if } 0 < x < D \\ h_S & \text{if } x = 0 \end{cases}$$

and

$$m(x) = \begin{cases} 0 & \text{if } x = L \\ m_D - M & \text{if } D < x < L \\ m_D & \text{if } x = D \\ m_S - M & \text{if } 0 < x < D \end{cases}$$

where M is a large constant. Clearly, if M is large enough, the manager will always choose to move the item when $x \neq D, L$. It is indeed only reasonable to keep units in storage at the warehouses and units that are in between warehouses will be moved down to the nearest downstream warehouse (or maybe even further) as quickly as possible. Also, the effective holding cost is equal to h_D if $D \leq x < L$ and to h_S if $0 \leq x < D$. Of course, there exist other alternatives that provide the same solution. For example one can include the actual moving cost in the holding cost and set $m(x) = 0$, so that the holding cost at the downstream location is smaller than the holding plus the moving cost between it and the upstream location.

The setting presented can be seen as an extension to Axsäter and Lundell [5]. The most significant difference between our model and theirs is that we have no restrictions on the holding cost apart from it being non-negative (in theirs it had to be increasing with x) and that the moving cost $m(x)$ also can depend on x and thus becomes a key driver of the inventory decision. Other recent work with continuous supply chains is Song and Zipkin [25], who assume that the holding cost is decreasing in x and consider zero moving cost $m(x)$. Another difference between our work and the ones cited above is that we focus on a discounted cost model whereas they consider average cost models. This minor change makes the formulation of the solution procedure much simpler. It allows us to characterize the optimal policy and derive closed-form solutions for some general problems.

3.2 The Formulation using the Unit-Decomposition Approach

In order to formally present the optimization problem, we use an observation that will simplify the exposition.

Lemma 1. *There exists an optimal policy such that units in the supply line never cross.*

This allows us to use the dynamic program formulation based on the single-unit tracking approach put forward by Axsäter [2]. This approach can indeed be used since order crossing is not optimal, all unmet demands are back-ordered and all costs are independent of what unit we are considering (they are linear per time-unit, per unit). The idea of single-unit tracking approach is to follow each item from the time it enters into the system (i.e., when it is ordered at $x = L$) until it exits (i.e., when it is used to satisfy customer demand at $x = 0$). One can hence monitor the cost associated with that unit and try to minimize the expected present value of this cost. This differs from the more traditional approach where one instead focuses on the inventory level, monitors its distribution and tries to minimize the expected cost associated with the evolution of this distribution.

Specifically, we define the cost-to-go function for unit k when it is located at x, denoted $J_k(x)$, as follows. Unit k is identified as the unit that will be used to serve the k-th next customer. That is, if there is currently a backlog of B customers waiting to be served, then it is the $(k + B)$-th unit of inventory in the chain, when ordering units in increasing order of x (i.e., it is the unit that will arrive to $x = 0$ in position $k + B$). Hence, we enumerate k so that 1 is the demand from the first customer that will arrive to the system counting from now and 2 the demand from the second customer counting from now, and so on. Consequently, $k \leq 0$ implies that unit k will be used to satisfy a demand that has already occurred. Figure 1 shows how the units are enumerated when there are $B = 3$ customers waiting for a product (if there are no customers waiting then the units will be enumerated 1, 2, 3, etc.).

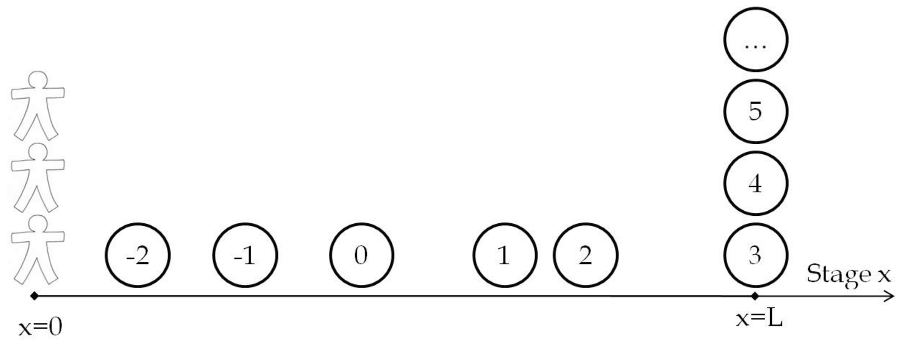


Figure 1: An example of a continuous supply chain. The x-axis represents the distance of each inventory unit from the customer. Each circle represents a unit of inventory. The number associated with its unit can be negative if the unit will serve a customer that has already arrived (there are $B = 3$ of them), or positive in which case it denotes the rank of the (future) customer to whom it will go.

$J_k(x)$ is defined as the minimum expected net present value of all back-order, holding and moving costs payed from now until that unit has been used to satisfy a demand from a customer. It of course depends upon where the unit is currently located, x, and what demand, measured by its rank k, it shall fulfill. For example, for $k \leq 0$, $J_k(x)$ is the net-present value of all back-order costs paid until that unit reaches the final customer plus all the moving and holding cost occurred while it is moved from stage x to stage 0. Note that in this case, the cost-to-go function $J_k(x)$ is identical for all $k \leq 0$, and for simplicity we will denote all these with $J_0(x)$. They can be expressed as:

$$J_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ \min\limits_{v \in \{0,1\}} \left\{ \Big(b + h(x) + vm(x)\Big)\Delta + J_0(x - v\Delta)e^{-r\Delta} \right\} & \text{otherwise} \end{cases} \tag{1}$$

where Δ is a short time interval.

If the demand has not occurred, i.e., $k \geq 1$, then the future costs depend upon when the customer arrives to the system and where the unit is at that moment. Since the demand is generated from a Poisson process, the time until the customer arrives is Erlang distributed with rate λ and index k, see e.g. Axsäter [2]. For the formulation here we only need to know that in a short time interval Δ, the probability of one customer arriving to the system is $\lambda\Delta$ and the probability of more customer arrivals is negligible, though. If a customer arrives, then the cost-to-go to be considered is the one corresponding to the $(k-1)$-th unit, rather than the k-th unit. The cost-to-go function can thus be expressed as

$$J_k(x) = \min_{v \in \{0,1\}} \left\{ \Big(h(x) + vm(x)\Big)\Delta + \Big((1 - \lambda\Delta)J_k(x - v\Delta) + \lambda\Delta J_{k-1}(x - v\Delta)\Big)e^{-r\Delta} \right\} \tag{2}$$

Of course, the derivation of Equations (1) and (2) is presented with a discrete formulation, with time increments of Δ. In reality, in a truly continuous-stage system, $J_k(x)$ satisfies a differential equation, called the Hamilton-Jacobi-Bellman (HJB) equation. The technical details from the continuous system are taken from optimal control theory, see Bertsekas [8]. The HJB equations can be written as

$$0 = \min_{v \in \{0,1\}} \left\{ v \left(m(x) - \frac{dJ_0}{dx} \right) + b + h(x) - rJ_0(x) \right\} \tag{3}$$

and for $k \geq 1$,

$$0 = \min_{v \in \{0,1\}} \left\{ v \left(m(x) - \frac{dJ_k}{dx} \right) + h(x) + \lambda J_{k-1}(x) - (\lambda + r)J_k(x) \right\} \tag{4}$$

Equations (3)-(4) are the counterparts of Equations (1)-(2) for the continuous-stage chain. Denote $v_k^*(x)$ the optimal control for unit k at location x.

The equations imply that it is optimal to move unit k forward, i.e., $v_k^*(x) = 1$, if and only if $m(x) \leq \frac{dJ_k}{dx}$, in which case

$$\frac{dJ_0}{dx} = b + h(x) + m(x) - rJ_0(x) \text{ and for } k \geq 1, \frac{dJ_k}{dx} = h(x) + m(x) + \lambda J_{k-1} - (\lambda + r)J_k. \quad (5)$$

Otherwise, $v_k^*(x) = 0$ and

$$J_0(x) = \frac{b + h(x)}{r} \text{ for } x > 0 \text{ and } J_k(x) = \frac{h(x) + \lambda J_{k-1}(x)}{r + \lambda} \text{ for } k \geq 1 \text{ and } x \geq 0. \quad (6)$$

If $v_0^*(x) = 0$ then no units at this location will be moved. As a result, if all units have to pass x to reach the end consumer then it will be optimal to never satisfy any demand using units that are located at x or further away. In essence, such a result implies that the cost of moving the unit from x or beyond to the consumer is higher than the cost of never servicing the customer. Hence, for the model to be realistic, L must be smaller than the smallest x where $v_0^*(x) = 0$. For completeness, we have the terminal condition $J_0(0) = 0$. Together, the equations above can provide a powerful scheme to obtain the optimal policy in many settings, as shown in the next section.

4 General Solution Procedure

In this section we will derive simple closed-form solutions for the optimal policy under various specific cost structures and provide a general procedure to find the solution under a general cost structure. We start by looking at the simplest case where the holding and moving costs are constant. We then analyze the case where they are linearly variable in x. Finally, we consider piece-constant costs, which can approximate general cost functions. We recover from the results that the optimal policy is an echelon base-stock policy, as shown by Muharremoglu and Tsitsiklis [21]. However, in contrast with most of the previous literature, we show that the optimal base-stock level is not necessarily increasing with x and the optimal steady-state policy will hence depend on L, i.e. the total distance to move a unit, as well.

4.1 Constant Costs

A constant cost model is obtained by setting $h(x) = h$ and $m(x) = m$ for all x, with h and m constant. For $k = 0$, it is simple to verify that J_0 can be expressed as

$$J_0(x) = \begin{cases} (b + h + m)\left(\frac{1 - e^{-rx}}{r}\right) & \text{if } e^{-rx} \geq \frac{m}{b + h + m} \\ \frac{b + h}{r} & \text{otherwise.} \end{cases} \quad (7)$$

using Equations (5)-(6) and the border condition $J_0(0) = 0$.

As a result, it is optimal to move the unit forward if and only if $x \le x_0^H$ where

$$x_0^H = \frac{1}{r} \ln \left(1 + \frac{b+h}{m}\right) \tag{8}$$

The result is intuitive: if the cost of moving the item is large, it is better to pay the holding and back-ordering costs indefinitely rather than incur the expense of moving the item. In that sense, if the upstream echelon is too "far" from the customer, i.e., $e^{-rL} < \dfrac{m}{b+h+m}$, then the manager simply chooses not to fulfill demand at all. Otherwise, it is optimal to ship the item downstream.

For $k \ge 1$, it turns out that the optimal decision is to move the unit if and only if x is within an interval $[x_k^L, x_k^H]$. Note that this is also true for $k = 0$ as well, with the lower bound of the interval being equal to $x_0^L = 0$. The optimal policy under this setting is expressed in the following theorem.

Theorem 1. Constant costs. *There exists a sequence of non-decreasing $x_k^L \ge 0$ and non-increasing $x_k^H \ge 0$, such that it is optimal to set $v_k^*(x) = 1$ if and only if $x_k^L \le x \le x_k^H$.*

This result is non-trivial. Indeed, it turns out that J_k is first convex and then concave. The proof relies on showing that $\dfrac{dJ_k}{dx}$ is first increasing and then decreasing (quasi-concave). While the proof is quite specific to the assumptions of constant costs, it can be extended to more general settings, as we will see later.

The thresholds x_k^L, x_k^H identified in the theorem completely characterize the optimal echelon base-stock levels.

Corollary 1. *The optimal echelon base-stock level at stage x is S if $x_S^L \le x < x_{S+1}^L$ or $x_{S+1}^H < x \le x_S^H$.*

The corollary above implies that the optimal policy is an echelon base-stock policy where the base-stock level, $S(x)$ is first non-decreasing in x as x_k^L is non-decreasing in k; it is then non-increasing in x as x_k^H is non-increasing in k. Note however that, in steady state, there will be no units launched from the factory with index larger than $S(L)$. Thus, in practice, even though at some $x < L$ the optimal echelon base-stock level might be strictly larger than $S(L)$, there will be no unit to be effectively moved. Thus, for any practical purpose, another set of base-stock levels that is non-decreasing in x may lead to the same inventory/shipping decisions, and hence to the same cost. In other words, they are also optimal. These two sets of echelon base-stocks are illustrated in Figure 2. In the figure, we can observe that is optimal to move inventory downstream in the bottom center region. It can be seen that for a given unit k, it is optimal to move the item if and only if $x_k^L \leq x \leq x_k^H$. For example for $k = 5$, the unit should be moved when $x_5^L = 4.17 \leq x \leq x_5^H = 5.31$. For $k \geq 6$, $x_k^L = \infty$ and $x_k^H = 0$, which implies that the item should not be moved, i.e., $v_6^*(x) = 0$ for all x, until a demand occurs and its rank change.

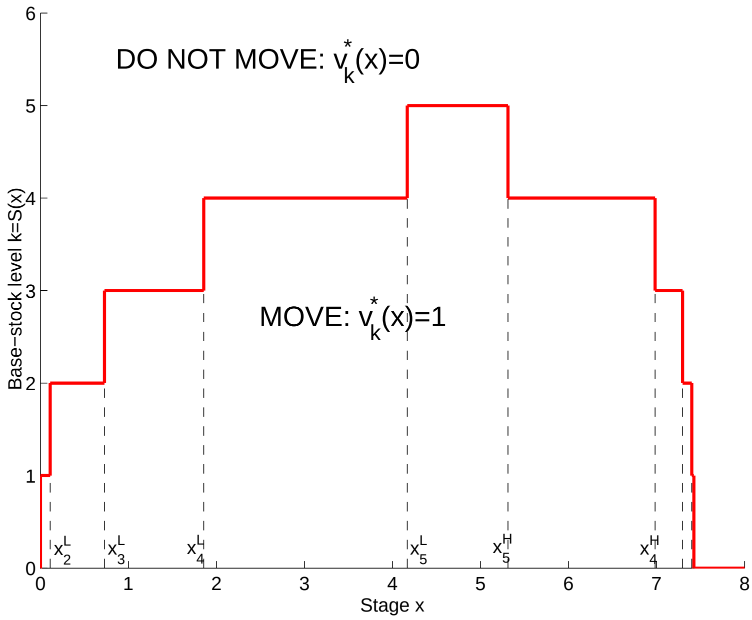


Figure 2: Optimal echelon base-stock levels $S(x)$ as a function of the stage x, with $b = 10, h = 1, m = 10$, and $b = 10, r = 0.1, \lambda = 1$.

It is worth noting that the traditional multi-echelon literature usually implies that the optimal echelon base-stock is non-decreasing as one moves upstream in the supply chain. In contrast, Corollary 1 reveals that it is possible that it decreases as one moves upstream. However, as earlier pointed out, in steady-state this is a possibility that never occurs, and hence a non-decreasing base-stock is also optimal. For instance, in Figure 2, if $L = 6$, $S(L) = 4$, and as a result, one will never see a 5-th unit in the supply chain (despite having $x_5^L = 4.17$ and $x_5^H = 5.31$, finite). Nevertheless, our result implies that there is a dependency between the upstream economics that determine $S(L)$, and the optimal base-stocks downstream. This implies that one cannot, generally, solve the problem by just finding the lower bound where a unit is stopped, x_k^L, but one must also find the upper bound where a unit is launched, x_k^H.

The key to finding the optimal policy is hence to find the threshold values x_k^L and x_k^H. In doing so, one can use the cost-to-go functions which can be determined from the following theorem.

Theorem 2. Constant costs. *The cost-to-go functions J_k can be expressed as*

$$J_k(x) = \begin{cases} \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } 0 \leq x \leq x_k^L \\ M_k - N_k e^{-rx} + \left(\displaystyle\sum_{i=0}^{k-1} A_{k,i} x^i\right) e^{-(r+\lambda)x} & \text{if } x_k^L \leq x \leq x_k^H \\ \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } x_k^H \leq x \end{cases}$$

where $M_k = \dfrac{h+m}{r} + \dfrac{b}{r}\left(\dfrac{\lambda}{\lambda + r}\right)^k$, $N_k = \dfrac{b+h+m}{r}$, $A_{k,i} = \dfrac{\lambda A_{k-1,i-1}}{i}$ *for* $i \geq 1$, *and* $A_{k,0}$ *such that* J_k *is continuous at* $x = x_k^L$.

The cost-to-go functions are hence easy to calculate when holding and moving costs are constant. Knowing them allows to calculate the thresholds easily. Indeed, given J_{k-1}, x_k^L is determined as the first x that fulfills $\dfrac{\lambda}{r+\lambda}\dfrac{dJ_{k-1}}{dx} = m$ which is numerically straightforward. Once x_k^L and hence $J_k\left(x_k^L\right)$ is known, one can use standard calculus to determine $A_{k,0}$. The only unknown that remains then is x_k^H, i.e. the first $x > x_k^L$ where again $\dfrac{dJ_k}{dx}$, corresponding to the solution to the HJB equation, is equal to m. Note that at this point $\dfrac{dJ_k}{dx}$ might be discontinuous because $\dfrac{\lambda}{r+\lambda}\dfrac{dJ_{k-1}}{dx}$ might be strictly less than m. This search can be simplified by using the fact that $x_k^L \leq x_k^H \leq x_{k-1}^H$. Finally, it should be noted that the first part of the procedure above that describes how to find x_k^L and $A_{k,0}$ is only valid for $k > 1$. For $k = 0$, we have already shown that $x_0^L = 0$ and $A_{0,0} = 0$. For $k = 1$, the value of x_1^L is either 0 or infinity as $\dfrac{dJ_0}{dx}$ is decreasing and $A_{1,0}$ can be found by using the border condition that $J_1(0) = \dfrac{h}{r+\lambda}$.

Figure 3 illustrates the value of $\dfrac{dJ_k}{dx}$ associated with the optimal policy from Figure 2. Note that the points at which $\dfrac{dJ_k}{dx} = m$ determine the thresholds x_k^L, x_k^H, see Figure 2.

4.2 Linear Costs

4.2.1 Linear Holding Costs

The results from §4.1 can be extended quite easily to situations where the holding cost is linearly increasing or decreasing with x. The proof is outlined below and follows the proof of Theorem 1. Consider $h(x) = h^0 + h'x$. To make the model reasonable, we assume that $h(x) > 0$ for all $x \in [0, L]$, which implies that $h^0 > 0$ and $L < -h^0/h'$ if the right-hand side is positive. We also assume that $m \geq \dfrac{h'}{r}$; otherwise, it would always be better to move the item towards $x = 0$ in order to save inventory costs, which would result in $v_k^*(x) = 1$ for all $k, x > 0$.

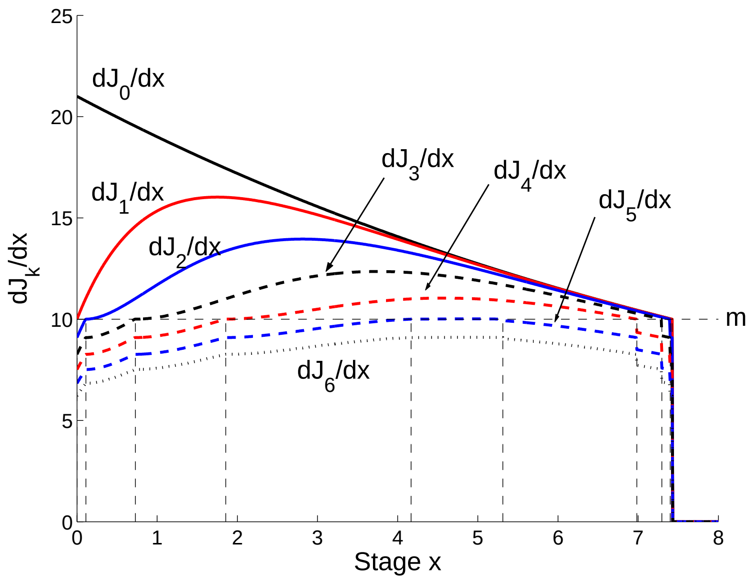


Figure 3: $\frac{dJ_k}{dx}$ as function of x, with $b = 10, h = 1, m = 10$, and $b = 10, r = 0.1, \lambda = 1$.

The cost-to-go function for $k = 0$ is then

$$J_0(x) = \begin{cases} \int_0^x (b + m + h^0 + h'(x-z))e^{-rz}dz = \\ \left(b + m + h^0 - \frac{h'}{r}\right)\frac{1-e^{-rx}}{r} + \frac{h'x}{r} & \text{if } e^{-rx} \geq \frac{m - h'/r}{m + b + h^0 - h'/r} \\ \frac{b + h^0 + h'x}{r} & \text{otherwise.} \end{cases}$$

The expression is very similar to Equation (7) but here takes into account the linear holding cost. The derivation relies on it always being optimal to move a demanded unit all the way to the customer, if it has been moved at all, which can be easily verified.

As is the case with constant costs $\frac{dJ_0}{dx}$ will be quasi-concave. Indeed, because $\frac{h'}{r} \leq m+b+h^0$, it is first decreasing down to m and then constant (there is a jump downward at x_0^H. We can similarly show quasi-concavity of $\frac{dJ_k}{dx}$ for $k \geq 1$. This leads to the following theorem.

Theorem 3. Linear holding costs. *When $h(x) = h^0 + h'x$, there exists a sequence of non-decreasing $x_k^L \geq 0$ and non-increasing $x_k^H \geq 0$, such that it is optimal to set $v_k^*(x) = 1$ if and* *only if* $x_k^L \le x \le x_k^H$*. The corresponding cost-to-go functions* J_k *can be expressed as*

$$J_k(x) = \begin{cases} \dfrac{h(x) + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } 0 \le x \le x_k^L \\ M_k - N_k e^{-rx} + B_k x + \left(\displaystyle\sum_{i=0}^{k-1} A_{k,i} x^i\right) e^{-(r+\lambda)x} & \text{if } x_k^L \le x \le x_k^H \\ \dfrac{h(x) + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } x_k^H \le x \end{cases} \tag{9}$$

where $M_k = \dfrac{h^0 + m - \frac{h'}{r}}{r} + \dfrac{b}{r}\left(\dfrac{\lambda}{\lambda + r}\right)^k$, $N_k = \dfrac{b + h^0 + m - \frac{h'}{r}}{r}$, $B_k = \dfrac{h'}{r}$, $A_{k,i} = \dfrac{\lambda A_{k-1,i-1}}{i}$ *for* $i \ge 1$*, and* $A_{k,0}$ *such that* J_k *is continuous at* $x = x_k^L$*.*

One can observe the similarity of this result with Theorems 1 and 2. Again, the cost-to-go functions can be calculated with this simple procedure, using that x_k^L is determined by

$$\frac{d}{dx}\left(\frac{h(x) + \lambda J_{k-1}(x)}{r+\lambda}\right) = m. \tag{10}$$

4.2.2 Linear Moving Costs

When the holding cost is constant and the moving cost is linear, the structure identified in §4.1 continues to exist, even though the analysis is not as easy as for the linear holding cost case for a number of reasons. Mainly, it is not sufficient that $\frac{dJ_k}{dx}$ is first increasing and then decreasing (quasi-concave), because $m(x)$ is evolving as well, and hence $\frac{dJ_k}{dx}(x)$ and $m(x)$ can cross more than twice even if the former is quasi-concave. Instead, we have to consider $F_k(x) = \frac{dJ_k}{dx}(x) - m(x)$. If this function is above 0 then it is optimal to move the unit and otherwise it is not. Unfortunately, this function $F_k(x)$ is not quasi-concave on the entire range, but it can be shown that it is quasi-concave for $x_k^L \le x \le x_{k-1}^H$ which is sufficient to guarantee the same structure of the optimal policy as the one described in §4.1.

We define $m(x) = m^0 + m'x$. To make the model reasonable, we assume that $m(x) > 0$ for all $x \in [0, L]$. The following theorem provides the extension of Theorems 1 and 2 to this case

Theorem 4. Linear moving costs. *When* $m(x) = m^0 + m'x$*, there exists a sequence of non-decreasing* $x_k^L \ge 0$ *and non-increasing* $x_k^H \ge 0$*, such that it is optimal to set* $v_k^*(x) = 1$ *if and only if* $x_k^L \le x \le x_k^H$*. The corresponding cost-to-go functions* J_k *can be expressed as*

$$J_k(x) = \begin{cases} \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } 0 \le x \le x_k^L \\ M_k - N_k e^{-rx} + B_k x + \left(\displaystyle\sum_{i=0}^{k-1} A_{k,i} x^i\right) e^{-(r+\lambda)x} & \text{if } x_k^L \le x \le x_k^H \\ \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } x_k^H \le x \end{cases} \tag{11}$$

where $M_k = \dfrac{h + m^0 - \frac{m'}{r}}{r} + \dfrac{b}{r}\left(\dfrac{\lambda}{\lambda + r}\right)^k$, $N_k = \dfrac{b + h + m^0 - \frac{m'}{r}}{r}$, $B_k = \dfrac{m'}{r}$, $A_{k,i} = \dfrac{\lambda A_{k-1,i-1}}{i}$ *for* $i \geq 1$, *and* $A_{k,0}$ *such that* J_k *is continuous at* $x = x_k^L$.

The proof of this result is quite different from the one of Theorem 3. However, it can be observed that the resulting expressions in Equation (11) have the exact same recursive form as the ones in (9), where h' is replaced by m'. One could think that the optimal policy would be the same if $h' = m'$. However, this is not so. Although the recursion is the same, the values obtained for x_k^L are not the same, since the conditions for determining them are different. With linear moving costs, the condition is

$$\frac{d}{dx}\left(\frac{h + \lambda J_{k-1}(x)}{r + \lambda}\right) = m(x).$$

instead of Equation (10). As a result, this changes the values of $A_{k,0}$, which results in different coefficients $A_{k,i}$.

4.3 Piecewise Constant Costs

Theorems 3 and 4 show that the structure of the problem remains the same as with constant costs when either holding or moving costs are linear. When there are general nonlinearities, the analysis quickly becomes intractable. In this section, we focus on the particular case of piecewise constant costs. This case is particularly interesting as it can provide an approximation for any holding and moving cost. Furthermore, such scenario is a good approximation of real supply chains. Indeed, inventory and production/distribution costs are locally stable and typically only exhibit changes at certain points. For instance, if we consider the supply chain of a shoe manufacturer, the inventory cost is relatively constant before manufacturing and after manufacturing; the cost of moving an item closer to the point of sales is also relatively constant, before the factory (e.g., shipping by truck), between factory and local distribution center (e.g., shipping by boat), and between local distribution center and store (e.g., shipping in a delivery van).

Before proceeding to the analysis, note that since now costs are discontinuous, then J_k might not be continuous. At the points of discontinuity, Equation (5) might not even be defined since the derivative may not exist. However, it is possible to still use Equations (3) and (4) everywhere except at the points of discontinuity. At the points of discontinuity, Equations (3) and (4) can be replaced with $J_0(x) = \min\left\{J_0(x^-), \dfrac{b + h(x^+)}{r}\right\}$ and $J_k(x) = \min\left\{J_k(x^-), \dfrac{h(x^+) + \lambda J_{k-1}(x^+)}{\lambda + r}\right\}$ for $k \geq 1$ where x^- denotes the vicinity of x

just before the change (limit below x) and x^+ just after it (limit above x). Hence the resulting cost-to-go functions J_k can only have jumps down.

As one can imagine, the structure derived above does not necessarily hold anymore. Since now there are several "regions" with different economics, there might exist more than one region where it is optimal to move the unit that shall satisfy the k-th demand. However, we can derive useful results to find the optimal policy and to describe the cost associated with this policy.

To illustrate the difficulties when costs are piecewise constant, consider the case where there is $z^1 = 0 \leq z^2$ such that

$$h(x) = \begin{cases} h^1 & \text{when } z^1 = 0 \leq x < z^2 \\ h^2 & \text{when } z^2 \leq x \end{cases}$$

and

$$m(x) = \begin{cases} m^1 & \text{when } z^1 = 0 \leq x < z^2 \\ m^2 & \text{when } z^2 \leq x \end{cases}$$

We use the variable $y^i = x - z^i$ rather than x and the cost-to-go function $J_k^i(y^i)$ expressed in this variable when $x \in [z^i, z^{i+1})$, i.e., $J_k(x) = J_k^i(y^i)$. We start with the decision for $k = 0$. Using a reformulated version of Equations (5) and (6), it can be shown that

$$J_0^i(y^i) = \begin{cases} M_0^i - N_0^i e^{-ry_i} & \text{if moved, i.e., if } e^{-ry_i} \geq \dfrac{m^i/r}{N_0^i} \\ \dfrac{b + h^i}{r} & \text{otherwise.} \end{cases}$$

where $M_0^i = \dfrac{b + h^i + m^i}{r}$ and N_0^i are such that the border condition $J_0^1(0) = 0$ and $J_0^2(0) = J_0^1(z^2 - z^1)$ is fulfilled. We already see that, if z^2 is large enough and $h^2 > h^1$, it is possible that $v_0^*(x) = 1$ if and only if $x_0^{L,1} = z^1 = 0 \leq x \leq x_0^{H,1} < z^2$ or $x_0^{L,2} = z^2 \leq x \leq x_0^{H,2}$. As a result, it is not optimal to move this item only in one interval, but in two. This shows that the structure identified in Theorems 1, 3 and 4 is not preserved.

More generally, let us denote the locations where the holding and/or moving cost changes $z^1 = 0, z^2, \ldots$. The corresponding costs in the segment $[z^i, z^{i+1})$ are denoted $h^1, h^2, \ldots$ and $m^1, m^2, \ldots$, respectively. Note that it is possible that $m^i = m^{i-1}$ if it is only the holding cost that changes at location z^i, or $h^i = h^{i-1}$.

With this notation, we can show that $\dfrac{dJ_0}{dx}$ thus quasi-concave in the segment $x \in [z^i, z^{i+1})$. Continuing along the lines of the proof of Theorem 1, this observation can be extended to $k \geq 0$: $\dfrac{dJ_k^i}{dy^i}$ is quasi-concave in its range. The following theorem shows the structure of the resulting optimal policy.

Theorem 5. Piecewise constant costs. *For each* $i = 1, 2, \ldots$, *there exists a sequence of non-decreasing* $x_k^{L,i} \geq 0$ *and non-increasing* $x_k^{H,i} \geq 0$, *such that, for* $z^i \leq x < z^{i+1}$, *it is optimal to set* $v_k^*(x) = 1$ *if and only if* $x_k^{L,i} \leq x \leq x_k^{H,i}$.

Unlike before with constant costs, the theorem shows that there might exist several regions where it is optimal to move the goods. Some conclusions can be drawn, though.

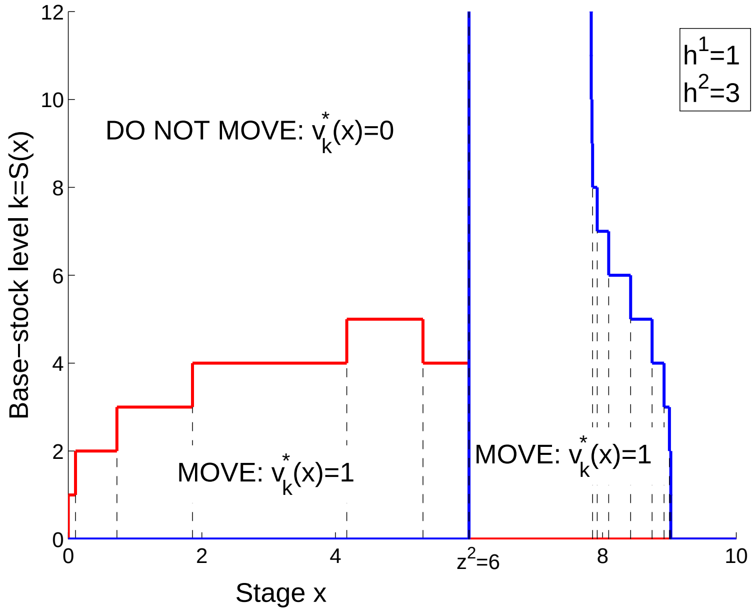
First, if the holding cost does not change, i.e., $h^{i-1} = h^i$, and if $v_k^*\left((z^i)^-\right) = 1$, then $v_k^*\left((z^i)^-\right) = v_k^*\left((z^i)^+\right) = 1$. This is true because the jump in $\frac{dJ_k}{dx}$ is the same as the jump in m at z^i, and hence $\frac{dJ_k}{dx} - m$ is smooth at z^i (so the optimal decision at $(z^i)^-$ is the same as at $(z^i)^+$). However, if $v_k^*\left((z^i)^-\right) = 0$, then it might be optimal to move it at $(z^i)^+$ even if it is not optimal to do so at $(z^i)^-$ if the moving cost decreases, but not if it increases. This is true because the change in $\frac{dJ_k}{dx}\left((z^i)^-\right) = \frac{\lambda}{\lambda + r}\frac{dJ_{k-1}^i}{dy^i}$ is smaller than the change in m at z^i and thus might be smaller than m^i in the former but not the later case.

Second, if it is the holding cost that changes but not the moving cost, then one might go from a move to a no-move decision, i.e., from $v_k^*\left((z^i)^-\right) = 1$ to $v_k^*\left((z^i)^+\right) = 0$, if $h^{i-1} < h^i$, but never the opposite. Vice-versa it may go from $v_k^*\left((z^i)^-\right) = 0$ to $v_k^*\left((z^i)^+\right) = 1$ if $h^{i-1} > h^i$, but never the opposite. Indeed, the changes in h will be reflected in the value of $\frac{dJ_k}{dx}$, directly if $v_k^*((z^i)^-) = 1$, or indirectly through the change in $\frac{dJ_{k-1}}{dx}$ if $v_k^*((z^i)^-) = 0$.

The policy identified in Theorem 5 is still similar to the one described in Corollary 1. It is illustrated in Figures 4 and 5. The figures use identical cost parameters in the range $z^1 = 0 \leq z \leq z^2 = 6$, which are the same costs as for Figure 2. Inventory costs are low compared to moving costs. As a result, since deferring the moving expense reduces the present value of that expense, the inventory levels are relatively low.

On the other hand, in the range $z^2 = 6 \leq x$, Figure 4 considers a situation where inventory costs are much higher. It is hence worth incurring the moving expense in order to take the inventory to the low-cost echelons, where they can be maintained cheaply. In the figure we also show the corresponding $\frac{dJ_k}{dx}$. We note that regardless of k, $\frac{dJ_k}{dx}((z^2)^+) > m$, which implies that in the vicinity of $z^2 = 6$, it is always optimal to move a unit, which suggests that $S((z^2)^+) = \infty$. In fact, one can show that when $(m^2 + h^2)\left(\frac{1 - e^{-r(x-z^2)}}{r}\right) + h^1\left(\frac{e^{-r(x-z^2)}}{r}\right) \leq \frac{h^2}{r}$, i.e., it is cheaper to move the unit downstream to z^2 and then hold forever, rather than hold forever at $x > z^2$, then $S(x) = \infty$.

In contrast, Figure 5 considers in the range $z^2 = 6 \leq x$ much lower moving costs. Thus the incentive to move the items downstream (so that they are available to customers sooner) is increased. As a result, the inventory levels, which where decreasing before z^2, start increasing



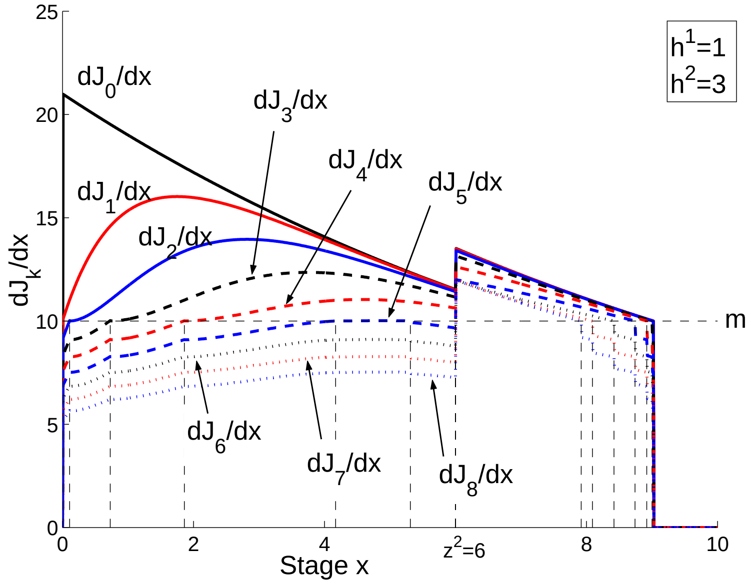


Figure 4: Optimal echelon base-stock levels $S(x)$ (top) and corresponding $\dfrac{dJ_k}{dx}$ (bottom) as a function of the stage x, with $z^1 = 0, z^2 = 6,\ b = 10, h^1 = 1, h^2 = 3, m^1 = m^2 = 10$, and $r = 0.1, \lambda = 1$.

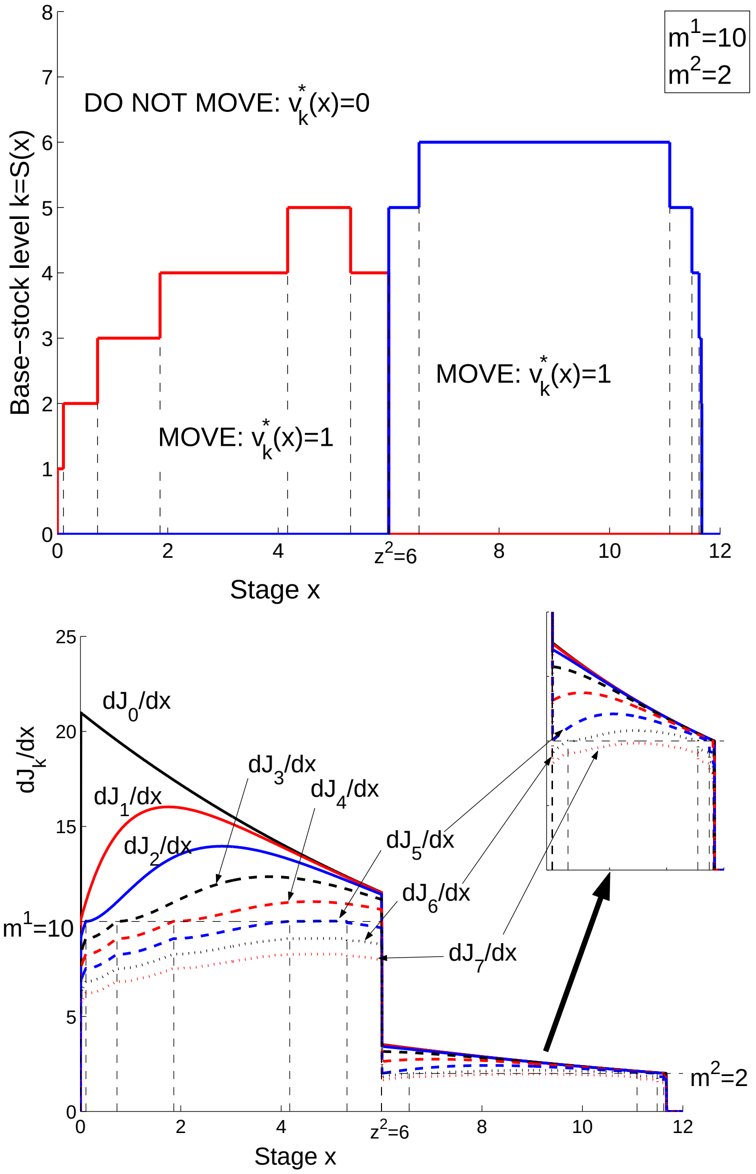


Figure 5: Optimal echelon base-stock levels $S(x)$ (top) and corresponding $\dfrac{dJ_k}{dx}$ (bottom) as a function of the stage x, with $z^1 = 0, z^2 = 6,\ b = 10, h^1 = h^2 = 1, m^1 = 10, m^2 = 2$, and $r = 0.1, \lambda = 1$.

again as one moves upstream: for instance, $S(8) = 6$, which is larger than the highest echelon base-stock level for $x \leq z^2$, 5. In this case, compared to Figure 4, the base-stock levels remain finite. In particular, $S((z^2)^+) < \infty$. Indeed, one can see in the corresponding $\frac{dJ_k}{dx}$ that, while both m and $\frac{dJ_k}{dx}$ exhibit a jump down at z^2, if $\frac{dJ_k}{dx} - m$ was positive at $(z^2)^-$, it stays positive at $(z^2)^+$, as we mentioned above.

4.4 General Cost Structures

In the general case, as seen in the previous section, the regions where $v_k^*(x) = 1$ might not be intervals. The procedure outlined above can still be used to solve Equations (3) and (4). Namely, one can find the optimal control v_k^* for $k = 0$ first, then for $k = 1$, and so on. For a given k, the procedure would be the following.

1. Calculate J_k^N identified by Equation (6).
2. Let $A_k = \{x | v_{k-1}^*(x) = 1\}$, which is made by a union of intervals $[x_{k-1}^{L,i}, x_{k-1}^{H,i}]$. We know that if $x \notin A_k$, then $v_k^*(x) = 0$, because at optimality orders do not cross.
3. Take an interval i. We know that $\frac{dJ_k^N}{dx}(x_{k-1}^{L,i}) < m(x_{k-1}^{L,i})$, which implies that $v_k^*(x_{k-1}^{L,i}) = 0$. Find the lowest $x > x_{k-1}^{L,i}$ such that $\frac{dJ_k^N}{dx}(x) = m(x)$, which we denote $x_k^{i_1,L}$. Hence for $x < x_k^{i_1,L}$, we know that $v_k^*(x) = 0$. After this point, solve Equation (5) until again $\frac{dJ_k^N}{dx}(x) = m(x)$, which occurs at $x_k^{i_1,H}$. In $[x_k^{i_1,L}, x_k^{i_1,H}]$, $v_k^*(x) = 1$. After $x_k^{i_1,H}$, find the next lowest $x > x_k^{i_1,H}$ such that $\frac{dJ_k^N}{dx}(x) = m(x)$: this determines $x_k^{i_2,L}$. Repeat the procedure above to find $x_k^{i_2,H}, x_k^{i_3,L}, x_k^{i_3,H}, \ldots$ until we reach $x_{k-1}^{H,i}$.
4. Repeat the step above for all intervals i, at which point we have determined v_k^* for all x.

This procedure is quite simple. However, it requires solving the differential HJB equation, Equation (5), which may be difficult if the functions h or m are complex. In the cases where these functions h and m are simple, e.g., polynomials, one can pre-compute the structure of the solutions of the differential equations, and hence the procedure in fact only involves finding the appropriate constants for the generic family of solutions to Equation (5). This shortcut is similar to the analytical expression of J_k derived in Theorems 2, 3 and 4. Computationally, the procedure was quite fast in all the instances that we used. However, for more complex structures of h and b, solving the differential equation numerically might be quite slow.

5 Impact of Costs on Inventory Placement

In this section, we are interested in using our analytical results to understand the relationship between costs and inventory levels. We first derive some analytical sensitivity results, and then complement these with an extensive numerical study.

5.1 Sensitivity Results

In order to derive analytical sensitivity results, we focus here on the constant cost scenario.

Theorem 6. *For $h(x) = h, m(x) = m$, for all k, x_k^L is non-increasing in b, h and non-decreasing in m; x_k^H is non-decreasing in b, h and non-increasing in m.*

The theorem derives how the thresholds x_k^L, x_k^H change with the model parameters. It is quite valuable since in most multi-echelon models sensitivity results are analytically intractable, and are only solved numerically. In contrast, our approach focuses on each unit separately, and this would allows us to determine how the thresholds vary. The same proof would even allow us to approximate the variation of the thresholds given parameter changes.

The insights of the theorem are quite intuitive. The impact of a higher back-ordering cost b is to increase inventory levels. The impact of a higher holding cost h is also to increase inventory levels. This might seem surprising, since for instance in the newsvendor model, the base-stock level decreases with h. However, one must keep in mind that, with constant costs, the inventory cost is paid *regardless* of the stage where the inventory is placed. As a result, as h increases, there is an incentive to sell the inventory quicker, which can be done by shifting more inventory downstream. In contrast, in the newsvendor model one only charges inventory costs at the lowest echelon. Finally, the impact of higher moving costs m is to reduce inventory levels. This is true because, when moving costs are higher, it is better to defer the moving expense (by doing so, the expense is discounted), and hence the inventory in the chain becomes smaller.

5.2 Numerical Experiments

We next illustrate numerically how the placement of inventory in the supply chain changes with costs. In particular, we explore how the thresholds change when costs are non constant, first for linear costs and then for piecewise-constant costs.

Figure 6 shows how the variation of h over the supply chain changes the echelon base-stock levels, which we represent through the thresholds x_k^L, x_k^H. The figure considers a linear holding cost $h(x) = h^0 + h'x$, for which Theorem 3 shows that the thresholds are unique.

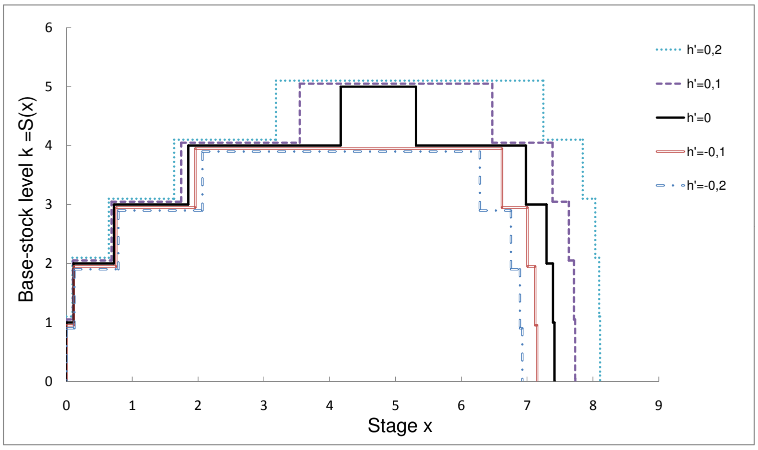


Figure 6: Variation of the thresholds x_k^L, x_k^H as a function of h', where $h(x) = h^0 + h'x$. The fixed part of the holding cost h^0 is chosen so that $h(5) = 1$ and the rest of the parameters are $b = 10, m = 10$, and $r = 0.1, \lambda = 1$.

As can be seen from the figure, an increase in h' implies larger inventory levels. In other words, the region where unit k should be moved is wider, for all k. For $h' > 0$ this is not at all surprising because, by moving a unit closer, one can reduce the holding cost as well as the expected future back-order cost which makes up for the fact that one has to pay the moving cost. The reduction in expected back-order cost is independent of the holding cost whereas the reduction in holding cost of course is increasing with h'. Thus, the higher h' is, the more beneficial it is to move a unit and this is true even if $h' < 0$ and thus the intuition behind the argument carries over to these values as well.

Figure 7 shows how the variation of m changes the thresholds x_k^L, x_k^H, using $m(x) = m^0 + m'x$. We can see that a decrease in m' reduces the inventory levels. This can be explained by the fact that for $x < 5$, a reduction in m' makes it more expensive to move the unit closer to the end consumer, and hence it is preferable to defer this cost to later, thereby reducing the echelon base-stock. This effect carries over to $x > 5$, because the sum of all future moving costs is decreasing with m', which diminishes the incentive of moving the unit, and reduces the inventory levels.

We now illustrate how the piecewise constant cost structure affects the optimal base-stock levels. For this purpose, we focus on the example shown in Figures 4 and 5, with $z^1 = 0, z^2 = 6$, and base parameters $b = 10, h^1 = h^2 = 1, m^1 = m^2 = 10$.

Figure 8 shows how the base-stock levels change with the value of h^2, keeping $m^2 = m^1 = 10$. It clearly shows that $x_k^{H,2}$ are non-decreasing in h^2, which is in line with the findings in Theorem

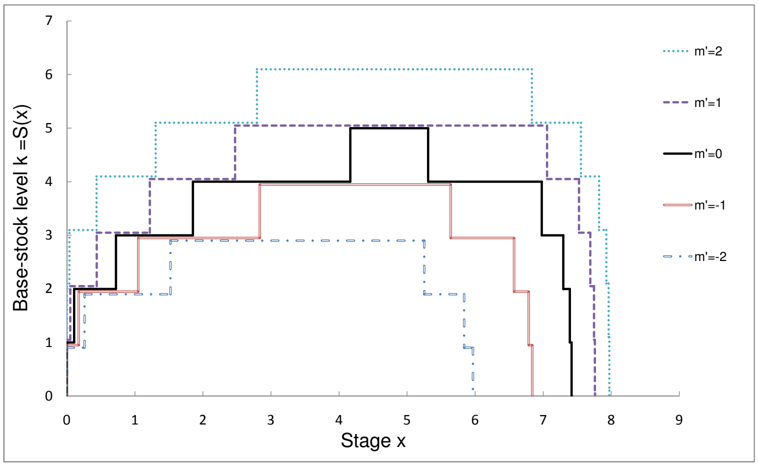


Figure 7: Variation of the thresholds x_k^L, x_k^H as a function of m', where $m(x) = m^0 + m'x$. The fixed part of the moving cost m^0 is chosen so that $m(5) = 10$ and the rest of the parameters are $b = 10, h = 1$, and $r = 0.1, \lambda = 1$.

6. In addition, we can see that when $h^2 \leq h^1$, then it not optimal to move units of rank 5 or higher for $x \geq z^2 = 6$ so the optimal base-stock level stays below $S((z^2)^-) = 4$. As soon as $h^2 > h^1$, then the base-stock level at $(z^2)^+$ shoots up to infinity. As discussed after Figure 4, we can show that $S(x) = \infty$, when $(m^2 + h^2)\left(\frac{1 - e^{-r(x-z^2)}}{r}\right) + h^1 \left(\frac{e^{-r(x-z^2)}}{r}\right) \leq \frac{h^2}{r}$. In other words, for all k, $x_k^{H,2} - z^2 \geq \frac{1}{r} \ln \left(1 + \frac{h^2 - h^1}{m^2}\right)$.

Similarly, Figure 9 shows the base-stock levels as a function of m^2, keeping $h^2 = h^1 = 1$. The insights of Theorem 6 are again verified as $x_k^{H,2}$ are non-increasing in the moving cost.

6 Conclusion and Further Research

In this paper, we have analyzed a continuous-stage multi-echelon inventory system. Under such setting, using the observation that under an optimal policy orders should not cross, we have decomposed the problem into a set of subproblems that can be solved one by one. The optimality conditions can be expressed through a set of HJB equations. In the case of constant or linear holding and moving costs, we characterize the structure of the solutions: it is optimal to ship the k-th item only for the stages in a given interval $[x_k^L, x_k^H]$. We also characterize the cost-to-go functions through simple expressions. For general costs, while the structure is no longer to ship a unit only in an interval, the solution procedure can still be applied to obtain

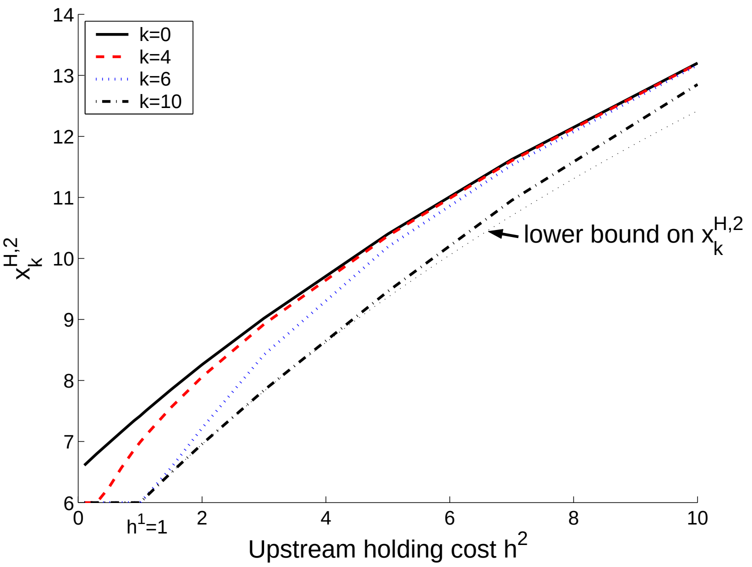


Figure 8: Upper thresholds $x_k^{H,2}$ as a function of the upstream holding cost h^2, with $z^1 = 0, z^2 = 6$, $b = 10, h^1 = 1, m^1 = m^2 = 10$, and $r = 0.1, \lambda = 1$.

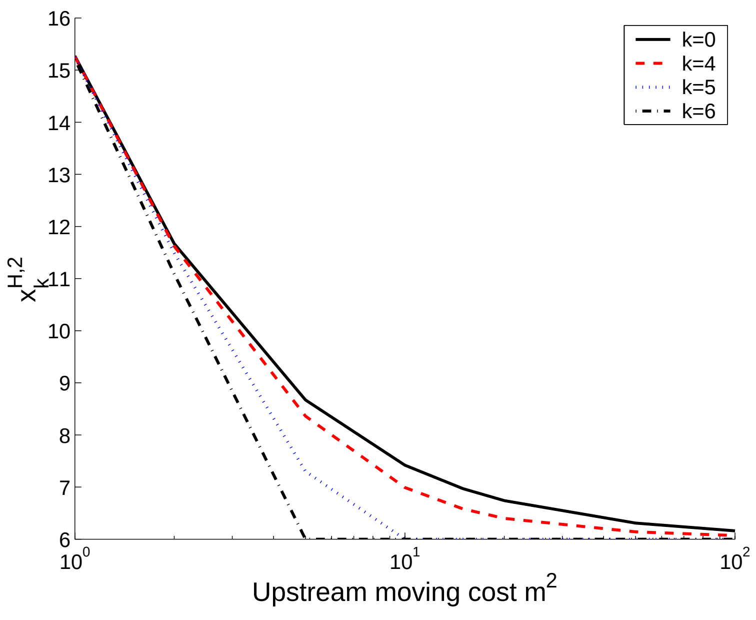


Figure 9: Upper thresholds $x_k^{H,2}$ as a function of the upstream moving cost m^2, with $z^1 = 0, z^2 = 6$, $b = 10, h^1 = h^2 = 1, m^1 = 10$, and $r = 0.1, \lambda = 1$.

numerical solutions. We finally study the sensitivity of the optimal echelon base-stock levels and find that they increase with higher back-ordering costs, higher holding costs (equal in all echelons), smaller difference between downstream and upstream holding costs, lower moving costs (equal in all echelons) and smaller difference between downstream and upstream moving costs. The approach presented here can thus be used to quickly find the optimal inventory placement in a multi-echelon system.

This work opens a number of research questions for further study. First, the general structure of the solutions can be used to generate heuristic inventory policies. These would complement the heuristics based on myopic decisions that are usually found from critical fractile ratios. Second, the assumption of Poisson demand could be relaxed, to include compound Poisson processes, or, more importantly, continuous demand processes, in which case the HJB equation would contain $\frac{\partial J}{\partial k}$ instead of $J_k - J_{k-1}$, similar to Song and Zipkin [25]. Third, the decision on moving or not an item could be enriched by considered several speeds. The model would hence find the optimal expediting decision. This is studied in The Authors [1]. Finally, distribution and assembly systems could also be considered, but these extensions present significant technical challenges.

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Appendix

Proof of Lemma 1

Proof. Consider an optimal policy and one sample path where two units cross. That is, unit 1 is ordered earlier than unit 2 (the time where it is moved at $x = L$ is strictly smaller for 1) but unit 2 arrives to $x = 0$ earlier than 1 (the time where unit 2 arrives at $x = 0$ is strictly smaller than for 1). Since the movement is continuous, if two units cross, consider the earliest time where they coincide in the same stage x. Since the moving and holding costs are independent of how stage x was reached, one can always choose to move unit 1 first, without changing the costs incurred. Consequently, order crossing cannot strictly reduce the cost, and a non-crossing policy is also optimal. ■

Proof of Theorem 1

Proof. We will prove the result by induction. The induction hypothesis is that, for $k \geq 0$, $\frac{dJ_k}{dx}$ is quasi-concave, first increasing and then decreasing (denote x_k^M the value at which it reaches the maximum), and that

$$J_k(x) = \begin{cases} \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } 0 \leq x \leq x_k^L \\ \text{satisfies HJB equation, (4)} & \text{if } x_k^L \leq x \leq x_k^H \\ \dfrac{h + \lambda J_{k-1}(x)}{r+\lambda} & \text{if } x_k^H \leq x \end{cases}$$

To initiate the induction, we use that for $k = 0$ $\frac{dJ_0}{dx}$ is quasi-concave because it is decreasing continuously for $x_0^L = x_0^M = 0 \leq x \leq x_0^H$ and then equal to zero for $x > x_0^H$.

Assume that the induction property is true for $k - 1 \geq 0$ and consider the problem for unit k. Let $J_k^Y(x)$ and $J_k^N(x) := \frac{h + \lambda J_{k-1}(x)}{r+\lambda}$ denote the the expected net present value of all costs if the unit is moved or not moved, respectively.

If it is optimal not to move unit k for all x, then $J_k = J_k^N$ for all x. This is true when $\frac{dJ_k}{dx} < m$ for all x. If this is true, then $\frac{dJ_k}{dx} = \frac{dJ_k^N}{dx} = \frac{\lambda}{r+\lambda}\frac{dJ_{k-1}}{dx}$. It is quasi-concave because $\frac{dJ_{k-1}}{dx}$ is quasi-concave from the induction property. In this case $x_k^L = \infty$ and $x_k^H = 0$.

In general, however, it may be optimal to move unit k for some x. We start showing that $\frac{dJ_k^N}{dx}$ is increasing before it is optimal to move the unit; that $\frac{dJ_k^Y}{dx}$ is then first increasing and then decreasing; and finally that $\frac{dJ_k^N}{dx}$ is decreasing once it is no longer optimal to move the unit again.

For $x \leq x_{k-1}^L$, since $\frac{dJ_{k-1}}{dx} \leq m$, then $\frac{dJ_k^N}{dx} \leq \frac{\lambda m}{r+\lambda} \leq m$, and hence it is optimal not to move the unit, i.e., $v_k^*(x) = 0$. Let $x_k^L := \min\left\{x \,\middle|\, \frac{dJ_k^N}{dx} \geq m\right\}$ (∞ if no such value exists, in which case there is nothing to show and then we can set $x_k^H = 0$). At $x = x_k^L$, $J_k = J_k^Y$, and thus solves the HJB equation

$$\frac{dJ_k^Y}{dx} = h + m + \lambda J_{k-1} - (\lambda + r)J_k^Y.$$

We claim that it is first increasing and then decreasing. Indeed, since J_{k-1} and J_k^Y are differentiable, then J_k^Y is infinitely differentiable. It turns out that $\frac{dJ_k^Y}{dx}$ is locally non-decreasing at $x = x_k^L$, because

$$\begin{aligned}\frac{d^2J_k^Y}{dx^2}(x_k^L) &= \lambda\frac{dJ_{k-1}}{dx}(x_k^L) - (\lambda+r)\frac{dJ_k^Y}{dx}(x_k^L)\\ &= (\lambda+r)\left(\frac{dJ_k^N}{dx}(x_k^L) - \frac{dJ_k^Y}{dx}(x_k^L)\right)\\ &= 0\end{aligned}$$

because $\frac{dJ_k^N}{dx}(x_k^L) = \frac{dJ_k^Y}{dx}(x_k^L) = m$. Furthermore, we have

$$\frac{d^3J_k^Y}{dx^3}(x_k^L) = \lambda\frac{dJ_{k-1}^2}{dx^2}(x_k^L) - (\lambda+r)\frac{d^2J_k^Y}{dx^2}(x_k^L) = \lambda\frac{dJ_{k-1}^2}{dx^2}(x_k^L) \geq 0$$

because we are in the convex part of J_{k-1}, i.e., $x \leq x_{k-1}^M$. Hence, $\frac{dJ_k^Y}{dx}$ is initially increasing above m.

Let $x_k^M := \min\left\{x \,\middle|\, x \geq x_k^L, \frac{d^2J_k^Y}{dx^2} = 0\right\}$. Because $\frac{dJ_k^Y}{dx}$ is initially increasing, this can only be a maximum. As a result,

$$\frac{d^3J_k^Y}{dx^3}(x_k^M) = \lambda\frac{d^2J_{k-1}}{dx^2}(x_k^M) - (\lambda+r)\frac{d^2J_k^Y}{dx^2}(x_k^M) = \lambda\frac{d^2J_{k-1}}{dx^2}(x_k^M) \leq 0.$$

This implies that $x_k^M \geq x_{k-1}^M$, i.e., x_k^M is in the region where J_{k-1} is concave. Hence, any point $x \geq x_k^M$ such that $\frac{d^2J_k^Y}{dx^2} = 0$ can only be a maximum too, which is impossible because for a differentiable function a maximum can only be followed by a minimum. Thus, $\frac{dJ_k^Y}{dx}$ is decreasing for $x \geq x_k^M$.

Denote $x_k^H := \min\left\{x \,\middle|\, x \geq x_k^M, \frac{dJ_k^Y}{dx} \leq m\right\}$ the value where $J_k^N(x)$ becomes smaller than the solution to J_k^Y again. At this point $J_k(x) = J_k^N(x)$ and $\frac{dJ_k^N}{dx} = \frac{\lambda}{r+\lambda}\frac{dJ_{k-1}}{dx} \leq m$. For $x \geq x_k^M \geq x_{k-1}^M$, $\frac{dJ_k^N}{dx}$ is decreasing because $\frac{dJ_{k-1}}{dx}$ is quasi-concave and decreasing after x_{k-1}^M. Furthermore, $x_k^H \leq x_{k-1}^H$ since $\frac{dJ_{k-1}}{dx}(x_k^H) = \frac{r+\lambda}{\lambda}\frac{dJ_k^N}{dx}(x_k^H) = \frac{r+\lambda}{\lambda}m \geq m$. This completes the induction. ■

Theorem 2

Proof. We can show it by induction. This is clearly true for $k = 0$. For $k \geq 1$, since $x_{k-1}^L \leq x_k^L \leq x \leq x_k^H \leq x_{k-1}^H$, J_k satisfies the HJB equation where J_{k-1} can be expressed as $M_k - N_k e^{-rx} + \left(\sum_{i=0}^{k-1} A_{k,i} x^i\right) e^{-(r+\lambda)x}$. This is a first-order differential equation that is solved with the parameters described in the theorem. ■

Theorem 3

Proof. We show quasi-concavity of $\frac{dJ_k}{dx}$ for $k \geq 0$, together with the structure of the cost-to-go function by induction. We have already shown it for $k = 0$ just before the theorem. Assume the induction property is true for $k - 1 \geq 0$.

As in the proof of Theorem 1, define the net present value of not moving a unit J_k^N

$$J_k^N(x) = \frac{h(x) + \lambda J_{k-1}(x)}{r + \lambda} = \frac{h^0 + h'x + \lambda J_{k-1}(x)}{r + \lambda}$$

This implies that

$$\frac{dJ_k^N}{dx}(x) = \frac{h' + \lambda \frac{dJ_{k-1}}{dx}(x)}{r + \lambda}$$

and hence

$$\frac{d^2 J_k^N}{dx^2}(x) = \frac{\lambda \frac{d^2 J_{k-1}}{dx^2}(x)}{r + \lambda}$$

It can be concluded that if $\frac{dJ_{k-1}}{dx}$ is quasi-concave, then $\frac{dJ_k^N}{dx}$ is quasi-concave too.

As a result, let $x_k^L := \min\left\{x \,\middle|\, \frac{dJ_k^N}{dx} \geq m\right\} \geq x_{k-1}^L$ (otherwise $\frac{dJ_k^N}{dx}(x_k^L) < m$). If $x_k^L < \infty$, then at some point it is optimal to start moving the unit forward. It can be noted that $x_k^L \leq x_{k-1}^M$, defined as the maximum of $\frac{dJ_{k-1}}{dx}$. At this point, we have

$$\frac{dJ_k}{dx}(x_k^L) = h(x_k^L) + m + \lambda J_{k-1}(x_k^L) - (r + \lambda) J_k(x_k^L) = m,$$

because J_{k-1} and J_k are continuous. Also,

$$\frac{d^2 J_k}{dx^2}(x_k^L) = h' + \frac{\lambda dJ_{k-1}}{dx}(x_k^L) - (r + \lambda)\frac{dJ_k}{dx}(x_k^L) = (r + \lambda)\left(\frac{dJ_k^N}{dx}(x_k^L) - \frac{dJ_k}{dx}(x_k^L)\right) = 0$$

and

$$\frac{d^3 J_k}{dx^3}(x_k^L) = \frac{\lambda d^2 J_{k-1}}{dx^2}(x_k^L) - (r + \lambda)\frac{d^2 J_k}{dx^2}(x_k^L) > 0$$

because at $x = x_k^L$, J_{k-1} is convex (otherwise, $\frac{dJ_k^N}{dx}$ could not be increasing).

Let $x_k^M := \min\left\{x \,\middle|\, x \geq x_k^L, \dfrac{d^2 J_k^Y}{dx^2} = 0\right\}$ be the (first) maximum point of $\dfrac{dJ_k}{dx}$. At this point we have

$$\frac{d^3 J_k}{dx^3}(x_k^M) = \frac{\lambda d^2 J_{k-1}}{dx^2}(x_k^M) - (r+\lambda)\frac{d^2 J_k}{dx^2}(x_M^L) < 0$$

which implies that $x_k^M > x_{k-1}^M$ and, as in the proof of Theorem 1, that $\dfrac{dJ_k}{dx}$ is decreasing after x_k^M. This results in having a $x_k^H := \min\left\{x \,\middle|\, x \geq x_k^M, \dfrac{dJ_k^Y}{dx} \leq m\right\} \leq x_{k-1}^H$ after which it is optimal not to move the inventory anymore.

To finish the induction proof, we can see that the cost-to-go functions verify Equation (9).

■

Proof of Theorem 4

Proof. The cost-to-go function for $k = 0$ can be expressed as

$$J_0(x) = \begin{cases} \int_0^x (b + m^0 + m'(x-z) + h)e^{-rz}dz = \\ \left(b + m^0 + h\right)\dfrac{1 - e^{-rx}}{r} + \dfrac{m'(e^{-rx} - 1 + rx)}{r^2} & \text{if } x \leq x_0^H \\ \dfrac{b+h}{r} & \text{otherwise.} \end{cases}$$

It is never optimal to move this unit only part of the way, as this only will result in a moving cost without any reduction in the holding or back-order cost. The function $F_0(x) := \dfrac{dJ_0}{dx} - m(x)$ is hence first increasing and then decreasing for $x_0^L = 0 \leq x \leq x_0^H$. The same is true for $\dfrac{dF_0}{dx}$, which is enough to initiate the induction proof. The induction property is that $F_k(x) := \dfrac{dJ_k}{dx} - m(x)$ and $\dfrac{dF_k}{dx}$ are quasi-concave first increasing and then decreasing, in $x_k^L \leq x \leq x_k^H$, for all $k \geq 0$. It is true for $k = 0$. Let us assume that it is true for $k-1$ and investigate the properties for k. We denote

$$F_k^N(x) := \frac{dJ_k^N}{dx}(x) - m(x) = \frac{\lambda \frac{dJ_{k-1}}{dx}(x)}{\lambda + r} - m(x) = \frac{\lambda F_{k-1}(x) - rm(x)}{\lambda + r},$$

and

$$F_k^Y(x) := \frac{dJ_k^Y}{dx}(x) - m(x) = h + \lambda J_{k-1}(x) - (\lambda + r)J_k(x).$$

If we differentiate these expression we obtain

$$\frac{dF_k^N}{dx}(x) = \frac{\lambda \frac{dF_{k-1}}{dx} - rm'}{\lambda + r}$$

and

$$\frac{dF_k^Y}{dx}(x) = \lambda F_{k-1}(x) - (\lambda + r)F_k(x) - rm(x) = (\lambda + r)\Big(F_k^N(x) - F_k(x)\Big).$$

Let x_k^L be the lowest x where $F_k^N(x) = 0$ and one goes from a no-move decision to a move decision. Observe that $x_k^L \geq x_{k-1}^L$ because $F_{k-1}(x_k^L) = 0$. At this point we have

$$\frac{dF_k^Y}{dx}(x_k^L) = (\lambda + r)\Big(m(x_k^L) - m(x_k^L)\Big) = 0$$

$$\frac{d^2 F_k^Y}{dx^2}(x_k^L) = \lambda \frac{dF_{k-1}}{dx}(x_k^L) - rm' = (\lambda + r)\frac{dF_k^N}{dx}(x_k^L).$$

The later expression must be greater than zero, unless $x_k^L = x_k^H$ (because F_k^N is increasing at x_k^L, as it crosses 0 from below). Let x_k^M be the first value were F_k^Y is maximized, i.e., where

$$\frac{dF_k^Y}{dx}(x_k^M) = 0$$

and

$$\frac{d^2 F_k^Y}{dx^2}(x_k^M) = \lambda \frac{dF_{k-1}}{dx}(x_k^M) - rm' < 0.$$

Since $\frac{dF_{k-1}}{dx}$ is quasi-concave in $[x_{k-1}^L, x_{k-1}^H]$ from the induction property, then because $\frac{dF_{k-1}}{dx}(x_k^M) < \frac{dF_{k-1}}{dx}(x_k^L)$, $\frac{dF_{k-1}}{dx}$ will continue to decrease after x_k^M as rm' is constant, and hence $\frac{d^2 F_k^Y}{dx^2} < 0$. Eventually, $F_k^Y(x) = 0$ at some point $x_k^H \leq x_{k-1}^H$. At this point $F_k^N(x_k^H) < 0$ and decreasing as $\frac{dF_k^N}{dx}(x_k^H) = \frac{1}{\lambda + r}\left(\lambda \frac{dF_{k-1}}{dx}(x_k^H) - rm'\right) < 0$ at least until $x = x_{k-1}^H$. For $x > x_{k-1}^H$, due to the non order-crossing property, $F_k^N(x) < 0$. Consequently, $F_k(x)$ will be quasi-concave for $x_k^L \leq x \leq x_k^H$.

It now only remains to show that $\frac{dF_k}{dx}(x)$ is quasi-concave for $x_k^L \leq x \leq x_k^H$. We know from above that $\frac{dF_k}{dx}(x)$ is increasing at x_k^L. Let $x_k^{M'}$ be the first x such that $\frac{d^2 F_k^Y}{dx^2}(x_k^{M'}) = 0$ and $\frac{d^3 F_k^Y}{dx^3}(x_k^{M'}) = \lambda \frac{d^2 F_{k-1}}{dx^2}(x_k^{M'}) < 0$, i.e., it is a maximum. It should be noted that $\frac{dF_{k-1}}{dx}(x)$ is decreasing at $x_k^{M'}$ and so it will be for all $\geq x_k^{M'}$ as well. As a result $\frac{dF_k^Y}{dx}(x)$ will be increasing for $x_k^L \leq x < x_k^{M'}$ and decreasing for $x_k^{M'} < x \leq x_k^H$. This completes the induction.

Finally, it can be easily verified that the expressions for the cost-to-go functions satisfy the HJB equation, (5). ∎

Proof of Theorem 5

Proof. In each segment $[z^i, z^{i+1})$, we apply the same proof as for Theorem 1, i.e., we show that $\frac{dJ_k^i}{dy^i}$ is quasi-concave. The only slight modification is that we restrict our attention to the segment, and hence the resulting thresholds satisfy the constraint that $x_k^{L,i} \geq z^i$ and $x_k^{H,i} \leq z^{i+1}$. ∎

Proof of Theorem 6

Proof. We show the theorem based on the proof of Theorem 1, by induction. Our induction hypothesis here is that, for $k \geq 0$, $\frac{d}{db}\left(\frac{dJ_k}{dx}\right) \geq 0$, $\frac{d}{dh}\left(\frac{dJ_k}{dx}\right) \geq 0$ and $\frac{d}{dm}\left(\frac{dJ_k}{dx}\right) \leq 1$. For $k \geq 1$, since x_k^L is the lowest x such that

$$\frac{dJ_k}{dx} - m \geq 0,$$

in a range where $\frac{dJ_k}{dx}$ is increasing, then increasing b or h or decreasing m reduces x_k^L. Similarly, since x_k^N is the lowest x higher than x_k^L such that

$$\frac{dJ_k}{dx} - m \leq 0,$$

in a range where $\frac{dJ_k}{dx}$ is decreasing, then increasing b or h or decreasing m induces x_k^H.

Let us show the induction property. For $k = 0$, we know that x_0^H increases with b and h and decreases with m. Equation (7) yields

$$\frac{dJ_0}{dx}(x) = \begin{cases} (b+h+m)e^{-rx} & \text{if } x \leq x_0^H \\ 0 & \text{otherwise.} \end{cases}$$

which satisfies the induction property. Assume it is true for $k-1 \geq 0$. For k, we have that J_k^N defined by Equation (6) satisfies $\frac{d}{db}\left(\frac{dJ_k^N}{dx}\right) \geq 0$, $\frac{d}{dh}\left(\frac{dJ_k^N}{dx}\right) \geq 0$ and $\frac{d}{dm}\left(\frac{dJ_k^N}{dx}\right) \leq 1$. This implies that increasing b or h or decreasing m reduces x_k^L. Consider now the derivative of the HJB equation, Equation (5):

$$\frac{d^2 J_k}{dx^2} = \lambda \frac{dJ_{k-1}}{dx} - (\lambda + r)\frac{dJ_k}{dx}$$

We claim that $\frac{d}{db}\left(\frac{dJ_k}{dx}\right) \geq 0$ in $[x_k^L, x_k^H]$. Indeed, consider in this interval the smallest x such that $\frac{d}{db}\left(\frac{dJ_k}{dx}\right) < 0$. For this x, $x > x_k^L$ because, as b increases, $\frac{dJ_k}{dx}(x_k^L)$ goes from m to a larger value. Using the induction property, the equation above implies that

$$\frac{d}{db}\left(\frac{d^2 J_k}{dx^2}\right) = \lambda \frac{d}{db}\left(\frac{dJ_{k-1}}{dx}\right) - (\lambda + r)\frac{d}{db}\left(\frac{dJ_k}{dx}\right) > 0.$$

This implies that $\frac{d}{db}\left(\frac{dJ_k}{dx}\right) < 0$ is increasing in x (derivatives can be exchanged), and hence this x is not the smallest x that satisfies the property. This is contradiction, and hence such x does not exist. The same argument shows that $\frac{d}{db}\left(\frac{dJ_k}{dx}\right) \geq 0$. For m, the argument is a little different. Consider in $[x_k^L, x_k^H]$ the smallest x such that $\frac{d}{dm}\left(\frac{dJ_k}{dx}\right) > 1$. Using the derivative of the HJB equation,

$$\frac{d}{dm}\left(\frac{d^2 J_k}{dx^2}\right) = \lambda \frac{d}{dm}\left(\frac{dJ_{k-1}}{dx}\right) - (\lambda + r)\frac{d}{dm}\left(\frac{dJ_k}{dx}\right) < 0.$$

This would imply that for a slightly smaller x, $\frac{d}{dm}\left(\frac{dJ_k}{dx}\right) > 1$ as well, which is a contradiction since at the low end of the interval $\frac{d}{dm}\left(\frac{dJ_k}{dx}\right)(x_k^L) \leq 1$. Hence $\frac{d}{dm}\left(\frac{dJ_k}{dx}\right) \leq 1$.

For $x > x_k^H$, we have already shown the effect on $\frac{dJ_k^N}{dx}$. This completes the proof of the induction property. ■